UNIVERSAL DERIVED EQUIVALENCES OF POSETS

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Abstract. By using only combinatorial data on two posets X and Y, we construct a set of so-called formulas. A formula produces simultaneously, for any abelian category $\mathcal{A}$, a functor between the categories of complexes of diagrams over X and Y with values in $\mathcal{A}$. This functor induces a triangulated functor between the corresponding derived categories.

This allows us to prove, for pairs X, Y of posets sharing certain common underlying combinatorial structure, that for any abelian category $\mathcal{A}$, regardless of its nature, the categories of diagrams over X and Y with values in $\mathcal{A}$ are derived equivalent.

1. Introduction

In previous work [3] we considered the question when the categories $\mathcal{A}^X$ and $\mathcal{A}^Y$ of diagrams over finite posets X and Y with values in the abelian category $\mathcal{A}$ of finite dimensional vector spaces over a fixed field k, are derived equivalent.

Since in that case the category of diagrams $\mathcal{A}^X$ is equivalent to the category of finitely generated modules over the incidence algebra kX, methods from the theory of derived equivalence of algebras, in particular tilting theory, could be used [2, 4, 5].

Interestingly, in all cases considered, the derived equivalence of two categories of diagrams does not depend on the field k. A natural question arises whether there is a general principle which explains this fact and extends to any arbitrary abelian category $\mathcal{A}$.

In this paper we provide a positive answer in the following sense; we exhibit several constructions of pairs of posets X and Y such that the derived categories $D(\mathcal{A}^X)$ and $D(\mathcal{A}^Y)$ are equivalent for any abelian category $\mathcal{A}$, regardless of its nature. Such pairs of posets are called *universally derived equivalent*, since the derived equivalence is universal and originates from the combinatorial and topological properties of the posets, rather than the specific abelian categories involved.

Our main tools are the so-called formulas. A formula consists of combinatorial data that produces simultaneously, for any abelian category $\mathcal{A}$, a functor between the categories of complexes of diagrams over X and Y with values in $\mathcal{A}$, which induces a triangulated functor between the corresponding derived categories.

1.1. The main construction.

Let X and Y be two finite partially ordered sets (*posets*). For $y \in Y$, write $[y, \cdot] = \{y' \in Y : y' \geq y\}$ and $[\cdot, y] = \{y' \in Y : y' \leq y\}$. Let $\{Y_x\}_{x \in X}$ be a collection of subsets of Y indexed by the elements of X, such that

$$[y, \cdot] \cap [y', \cdot] = \phi \quad \text{and} \quad [\cdot, y] \cap [\cdot, y'] = \phi \tag{1.1}$$

for any $x \in X$ and $y \neq y'$ in Y_x. Assume in addition that for any $x \leq x'$, there exists an isomorphism $\varphi_{x,x'} : Y_x \xrightarrow{\sim} Y_{x'}$ such that

$$y \leq \varphi_{x,x'}(y) \qquad \text{for all } y \in Y_x \tag{1.2}$$

By (1.1), it follows that

$$\varphi_{x,x''} = \varphi_{x',x''}\varphi_{x,x'} \qquad \text{for all } x \leq x' \leq x''. \tag{1.3}$$

Define two partial orders $\leq_+$ and $\leq_-$ on the disjoint union $X \sqcup Y$ as follows. Inside X and Y, the orders $\leq_+$ and $\leq_-$ agree with the original ones, and for $x \in X$ and $y \in Y$ we set

$$\begin{aligned} x \leq_+ y &\iff \exists\, y_x \in Y_x \text{ with } y_x \leq y \\ y \leq_- x &\iff \exists\, y_x \in Y_x \text{ with } y \leq y_x \end{aligned} \tag{1.4}$$

with no other relations (note that the element y_x is unique by (1.1), and that $\leq_+$, $\leq_-$ are partial orders by (1.2)).

Theorem 1.1. *The two posets* $(X \sqcup Y, \leq_+)$ *and* $(X \sqcup Y, \leq_-)$ *are universally derived equivalent.*

The assumption (1.1) of the Theorem cannot be dropped, as demonstrated by the following example.

Example 1.2. Consider the two posets whose Hasse diagrams are given by

$(X \sqcup Y, \leq_+)$ $\qquad$ $(X \sqcup Y, \leq_-)$

They can be represented as $(X \sqcup Y, \leq_+)$ and $(X \sqcup Y, \leq_-)$ where $X = \{1\}$, $Y = \{2, 3, 4\}$ and $Y_1 = \{2, 3\} \subset Y$. The categories of diagrams over these two posets are in general not derived equivalent, even for diagrams of vector spaces.

The construction of Theorem 1.1 has many interesting consequences, some of them related to ordinal sums and others to generalized BGP reflections [1]. First, consider the case where all the subsets Y_x are single points, that is, there exists a function $f : X \to Y$ with $Y_x = \{f(x)\}$ for all $x \in X$. Then (1.1) and (1.3) are automatically satisfied and the condition (1.2) is equivalent to f being *order preserving*, i.e. $f(x) \leq f(x')$ for $x \leq x'$. Let $\leq^f_+$ and $\leq^f_-$ denote the corresponding orders on $X \sqcup Y$, and note that (1.4) takes the simplified form

$$\begin{aligned} x \leq^f_+ y &\iff f(x) \leq y \\ y \leq^f_- x &\iff y \leq f(x) \end{aligned} \tag{1.5}$$

Corollary 1.3. *Let* $f : X \to Y$ *be order preserving. Then the two posets* $(X \sqcup Y, \leq^f_+)$ *and* $(X \sqcup Y, \leq^f_-)$ *are universally derived equivalent.*

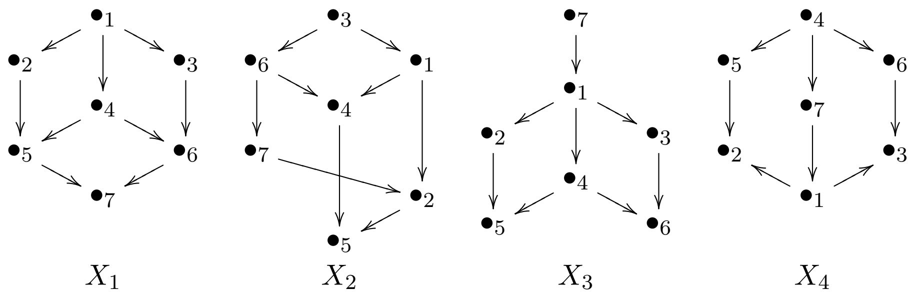

FIGURE 1. Four universally derived equivalent posets

Example 1.4. Consider the four posets X_1, X_2, X_3, X_4 whose Hasse diagrams are drawn in Figure 1. For any of the pairs (i,j) where $(i,j) = (1,2)$, $(1,3)$ or $(3,4)$ we find posets X_{ij} and X_{ji} and an order-preserving function $f_{ij} : X_{ij} \to X_{ji}$ such that

$$X_i \simeq (X_{ij} \sqcup X_{ji}, \leq_{+}^{f_{ij}}) \qquad X_j \simeq (X_{ij} \sqcup X_{ji}, \leq_{-}^{f_{ij}})$$

hence X_i and X_j are universally derived equivalent. Indeed, let

$$X_{12} = \{1,2,4,5\} \qquad X_{21} = \{3,6,7\}$$
$$X_{13} = \{1,2,3,4,5,6\} \qquad X_{31} = \{7\}$$
$$X_{34} = \{1,2,3,7\} \qquad X_{43} = \{4,5,6\}$$

and define $f_{12} : X_{12} \to X_{21}$, $f_{13} : X_{13} \to X_{31}$ and $f_{34} : X_{34} \to X_{43}$ by

$$f_{12}(1) = 3 \qquad f_{12}(2) = f_{12}(5) = 7 \qquad f_{12}(4) = 6$$
$$f_{13}(1) = \cdots = f_{13}(6) = 7$$
$$f_{34}(1) = f_{34}(7) = 4 \qquad f_{34}(2) = 5 \qquad f_{34}(3) = 6$$

1.2. **Applications to ordinal sums.** Recall that the *ordinal sum* of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$, denoted $P \oplus Q$, is the poset $(P \sqcup Q, \leq)$ where $x \leq y$ if $x, y \in P$ and $x \leq_P y$ or $x, y \in Q$ and $x \leq_Q y$ or $x \in P$ and $y \in Q$. Similarly, the *direct sum* $P + Q$ is the poset $(P \sqcup Q, \leq)$ where $x \leq y$ if $x, y \in P$ and $x \leq_P y$ or $x, y \in Q$ and $x \leq_Q y$. Note that the direct sum is commutative (up to isomorphism) but the ordinal sum is not. Denote by $\mathbf{1}$ the poset consisting of one element. Taking $Y = \mathbf{1}$ in Corollary 1.3, we get the following

Corollary 1.5. *For any poset X, the posets $X \oplus \mathbf{1}$ and $\mathbf{1} \oplus X$ are universally derived equivalent.*

Note that for arbitrary two posets X and Y, it is true that for any field k, the categories of diagrams of finite dimensional k-vector spaces over $X \oplus Y$ and $Y \oplus X$ are derived equivalent [3, Corollary 4.15]. However the proof relies on the notion of tilting complexes and cannot be directly extended to arbitrary abelian categories.

In Section 4.3 we prove the following additional consequence of Corollary 1.3 for ordinal and direct sums.

Corollary 1.6. *For any two posets X and Z, the posets $X \oplus \mathbf{1} \oplus Z$ and $\mathbf{1} \oplus (X + Z)$ are universally derived equivalent. Hence the posets $X \oplus \mathbf{1} \oplus Z$ and $Z \oplus \mathbf{1} \oplus X$ are universally derived equivalent.*

The result of Corollary 1.6 is no longer true when $\mathbf{1}$ is replaced by an arbitrary poset, even for diagrams of vector spaces, see [3, Example 4.20].

1.3. **Generalized BGP reflections.** More consequences of Theorem 1.1 are obtained by considering the case where $X = \{*\}$ is a single point, that is, there exists a subset $Y_0 \subseteq Y$ such that (1.1) holds for any $y \neq y'$ in Y_0. Observe that conditions (1.2) and (1.3) automatically hold in this case, and the two partial orders on $Y \cup \{*\}$ corresponding to (1.4), denoted $\le_+^{Y_0}$ and $\le_-^{Y_0}$, are obtained by extending the order on Y according to

$$
\begin{aligned}
* <_+^{Y_0} y &\iff \exists\, y_0 \in Y_0 \text{ with } y_0 \le y \\
y <_-^{Y_0} * &\iff \exists\, y_0 \in Y_0 \text{ with } y \le y_0
\end{aligned}
\tag{1.6}
$$

Corollary 1.7. *Let $Y_0 \subseteq Y$ be a subset satisfying* (1.1)*. Then the posets $(Y \cup \{*\}, \le_+^{Y_0})$ and $(Y \cup \{*\}, \le_-^{Y_0})$ are universally derived equivalent.*

Note that in the Hasse diagram of $\le_+^{Y_0}$, the vertex $*$ is a source which is connected to the vertices of Y_0, and the Hasse diagram of $\le_-^{Y_0}$ is obtained by reverting the orientations of the arrows from $*$, making it into a sink. Thus Corollary 1.7 can be considered as a generalized BGP reflection principle.

Viewing orientations on (finite) trees as posets by setting $x \le y$ for two vertices x, y if there exists an oriented path from x to y, and applying a standard combinatorial argument [1], we recover the following corollary, already known for categories of vector spaces over a field.

Corollary 1.8. *Any two orientations of a tree are universally derived equivalent.*

1.4. **Formulas.** By using only combinatorial data on two posets X and Y, we construct a set of formulas $\mathcal{F}_X^Y$. A formula $\boldsymbol{\xi}$ produces simultaneously, for any abelian category $\mathcal{A}$, a functor $F_{\boldsymbol{\xi},\mathcal{A}}$ between the categories $C(\mathcal{A}^X)$ and $C(\mathcal{A}^Y)$ of complexes of diagrams over X and Y with values in $\mathcal{A}$. This functor induces a triangulated functor $\widetilde{F}_{\boldsymbol{\xi},\mathcal{A}}$ between the corresponding derived categories $D(\mathcal{A}^X)$ and $D(\mathcal{A}^Y)$ such that the following diagram is commutative

$$
\begin{array}{ccc}
C(\mathcal{A}^X) & \xrightarrow{F_{\boldsymbol{\xi},\mathcal{A}}} & C(\mathcal{A}^Y) \\
\downarrow & & \downarrow \\
D(\mathcal{A}^X) & \xrightarrow{\widetilde{F}_{\boldsymbol{\xi},\mathcal{A}}} & D(\mathcal{A}^Y)
\end{array}
$$

where the vertical arrows are the canonical localizations.

We prove Theorem 1.1 by exhibiting a pair of formulas $\boldsymbol{\xi}^+ \in \mathcal{F}_{\le_+}^{\le_-}$, $\boldsymbol{\xi}^- \in \mathcal{F}_{\le_-}^{\le_+}$ and showing that for any abelian category $\mathcal{A}$, the compositions $\widetilde{F}_{\boldsymbol{\xi}^+,\mathcal{A}} \widetilde{F}_{\boldsymbol{\xi}^-,\mathcal{A}}$ and $\widetilde{F}_{\boldsymbol{\xi}^-,\mathcal{A}} \widetilde{F}_{\boldsymbol{\xi}^+,\mathcal{A}}$ of the corresponding triangulated functors on the derived categories are auto-equivalences, as they are isomorphic to the translations. Hence $\le_+$ and $\le_-$ are universally derived equivalent.

2. Complexes of diagrams

2.1. **Diagrams and sheaves.** Let X be a poset and let $\mathcal{A}$ be a category.

Definition 2.1. A *diagram* (A, r) over X with values in $\mathcal{A}$ consists of the following data:

- For any $x \in X$, an object A_x of $\mathcal{A}$
- For any pair $x \leq x'$, a morphism $r_{xx'} : A_x \to A_{x'}$ (*restriction map*)

subject to the conditions $r_{xx} = \mathrm{id}_{A_x}$ and $r_{xx''} = r_{x'x''} r_{xx'}$ for all $x \leq x' \leq x''$ in X.

A *morphism* $f : (A, r) \to (A', r')$ of diagrams consists of morphisms $f_x : A_x \to A'_x$ for all $x \in X$, such that for any $x \leq x'$, the diagram

$$\begin{array}{ccc} A_x & \xrightarrow{f_x} & A'_x \\ {\scriptstyle r_{xx'}}\downarrow & & \downarrow{\scriptstyle r'_{xx'}} \\ A_{x'} & \xrightarrow[f_{x'}]{} & A'_{x'} \end{array}$$

commutes.

Using these definitions, we can speak of the category of diagrams over X with values in $\mathcal{A}$, which will be denoted by $\mathcal{A}^X$.

We can view X as a small category as follows. Its objects are the points $x \in X$, while $\mathrm{Hom}_X(x, x')$ is a one-element set if $x \leq x'$ and empty otherwise. Under this viewpoint, a diagram over X with values in $\mathcal{A}$ becomes a functor $A : X \to \mathcal{A}$ and a morphism of diagrams corresponds to a natural transformation, so that $\mathcal{A}^X$ is naturally identified with the category of functors $X \to \mathcal{A}$. Observe that any functor $F : \mathcal{A} \to \mathcal{A}'$ induces a functor $F^X : \mathcal{A}^X \to \mathcal{A}'^X$ by the composition $F^X(A) = F \circ A$. In terms of diagrams and morphisms, $F^X(A, r) = (FA, Fr)$ where $(FA)_x = F(A_x)$, $(Fr)_{xx'} = F(r_{xx'})$ and $F^X(f)_x = F(f_x)$.

If $\mathcal{A}$ is additive, then $\mathcal{A}^X$ is additive. Assume now that $\mathcal{A}$ is abelian. In this case, $\mathcal{A}^X$ is also abelian, and kernels, images, and quotients can be computed pointwise, that is, if $f : (A, r) \to (A', r')$ is a morphism of diagrams then $(\ker f)_x = \ker f_x$, $(\mathrm{im}\, f)_x = \mathrm{im}\, f_x$, with the restriction maps induced from r, r'. In particular, for any $x \in X$ the evaluation functor $-_x : \mathcal{A}^X \to \mathcal{A}$ taking a diagram (A, r) to A_x and a morphism $f = (f_x)$ to f_x, is exact.

The poset X admits a natural topology, whose open sets are the subsets $U \subseteq X$ with the property that if $x \in U$ and $x \leq x'$ then $x' \in U$. The category of diagrams over X with values in $\mathcal{A}$ can then be naturally identified with the category of sheaves over the topological space X with values in $\mathcal{A}$ [3].

2.2. **Complexes and cones.** Let $\mathcal{B}$ be an additive category. A *complex* $(K^\bullet, d_K^\bullet)$ over $\mathcal{B}$ consists of objects K^i for $i \in \mathbb{Z}$ with morphisms $d_K^i : K^i \to K^{i+1}$ such that $d_K^{i+1} d_K^i = 0$ for all $i \in \mathbb{Z}$. If $n \in \mathbb{Z}$, the *shift* of $K^\bullet$ by n, denoted $K[n]^\bullet$, is the complex defined by $K[n]^i = K^{i+n}$, $d_{K[n]}^i = (-1)^n d_K^{i+n}$.

Let $(K^\bullet, d_K^\bullet)$, $(L^\bullet, d_L^\bullet)$ be two complexes and $f = (f^i)_{i \in \mathbb{Z}}$ a collection of morphisms $f^i : K^i \to L^i$. If $n \in \mathbb{Z}$, let $f[n] = (f[n]^i)_{i \in \mathbb{Z}}$ with $f[n]^i = f^{i+n}$.

Using this notation, the condition that f is a *morphism* of complexes is expressed as $f[1]d_K = d_L f$.

The *cone* of a morphism $f : K^\bullet \to L^\bullet$, denoted $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)$, is the complex whose i-th entry equals $K^{i+1} \oplus L^i$, with the differential

$$d(k^{i+1}, l^i) = (-d_K^{i+1}(k^{i+1}), f^{i+1}(k^{i+1}) + d_L^i(l^i))$$

In a more compact form, $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet) = K[1]^\bullet \oplus L^\bullet$ with the differential acting as the matrix

$$\begin{pmatrix} d_K[1] & 0 \\ f[1] & d_L \end{pmatrix}$$

by viewing the entries as column vectors.

When $\mathcal{B}$ is abelian, the *i-th cohomology* of $(K^\bullet, d_K^\bullet)$ is defined by $\mathrm{H}^i(K^\bullet) = \ker d_K^i / \operatorname{im} d_K^{i-1}$, and $(K^\bullet, d_K^\bullet)$ is *acyclic* if $\mathrm{H}^i(K^\bullet) = 0$ for all $i \in \mathbb{Z}$. A morphism $f : K^\bullet \to L^\bullet$ induces morphisms $\mathrm{H}^i(f) : \mathrm{H}^i(K^\bullet) \to \mathrm{H}^i(L^\bullet)$. f is called a *quasi-isomorphism* if $\mathrm{H}^i(f)$ are isomorphisms for all $i \in \mathbb{Z}$.

The following lemma is standard.

Lemma 2.2. *$f : K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if the cone $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)$ is acyclic.*

Let $C(\mathcal{B})$ denote the category of complexes over $\mathcal{B}$. Denote by $[1] : C(\mathcal{B}) \to C(\mathcal{B})$ the shift functor taking a complex $(K^\bullet, d_K^\bullet)$ to $(K[1]^\bullet, d_{K[1]^\bullet})$ and a morphism f to $f[1]$. Any additive functor $G : \mathcal{B} \to \mathcal{B}'$ induces an additive functor $C(G) : C(\mathcal{B}) \to C(\mathcal{B}')$ by sending a complex $((K^i), (d_K^i))$ to $((G(K^i)), (G(d_K^i)))$ and a morphism (f^i) to $(G(f^i))$.

Lemma 2.3. *For any additive category $\mathcal{A}$ and a poset X, there exists an equivalence of categories $\Phi_{X,\mathcal{A}} : C(\mathcal{A}^X) \simeq C(\mathcal{A})^X$ such that for any additive category $\mathcal{A}'$ and an additive functor $F : \mathcal{A} \to \mathcal{A}'$, the diagram*

$$\begin{array}{ccc} C(\mathcal{A})^X & \xrightarrow[\sim]{\Phi_{X,\mathcal{A}}} & C(\mathcal{A})^X \\ {\scriptstyle C(F^X)}\downarrow & & \downarrow{\scriptstyle C(F)^X} \\ C(\mathcal{A}')^X & \xrightarrow[\sim]{\Phi_{X,\mathcal{A}'}} & C(\mathcal{A}')^X \end{array} \tag{2.1}$$

commutes. In other words, we can identify a complex of diagrams with a diagram of complexes.

Proof. Let $\mathcal{A}$ be additive and let $(K^\bullet, d^\bullet)$ be a complex in $C(\mathcal{A}^X)$. Denote by $d^i : K^i \to K^{i+1}$ the morphisms in $\mathcal{A}^X$ and by $d_x^i : K_x^i \to K_x^{i+1}$ the morphisms on the stalks. Let $r_{xy}^i : K_x^i \to K_y^i$ denote the restriction maps in the diagram K^i.

For a morphism $f : (K^\bullet, d^\bullet) \to (L^\bullet, d^\bullet)$ in $C(\mathcal{A}^X)$, denote by $f^i : K^i \to L^i$ the corresponding morphisms in $\mathcal{A}^X$ and by $f_x^i : K_x^i \to L_x^i$ the morphisms on stalks. Define a functor $\Phi : C(\mathcal{A}^X) \to C(\mathcal{A})^X$ by

$$\Phi_{X,\mathcal{A}}(K^\bullet, d^\bullet) = (\{K_x^\bullet\}_{x \in X}, \{r_{xy}\}) \qquad \Phi_{X,\mathcal{A}}(f) = (f_x)_{x \in X}$$

where $(K_x^\bullet)^i = K_x^i$ with differential $d_x^\bullet = (d_x^i)^i$, $r_{xy} = (r_{xy}^i)^i : K_x^\bullet \to K_y^\bullet$ are the restriction maps, and $f_x = (f_x^i)^i : K_x^\bullet \to L_x^\bullet$.

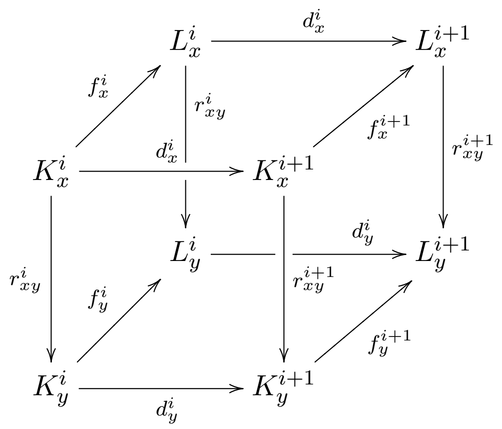

FIGURE 2

The commutativity of all squares in the diagram in Figure 2 implies that $\Phi_{X,\mathcal{A}}$ is well-defined, induces the required equivalence and that (2.1) commutes. $\square$

In the sequel, X is a poset, $\mathcal{A}$ is an abelian category and all complexes are in $C(\mathcal{A}^X)$.

Lemma 2.4. $\mathrm{H}^i(K^\bullet)_x = \mathrm{H}^i(K^\bullet_x)$

Proof. Kernels and images can be computed pointwise. $\square$

Lemma 2.5. $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)_x = \mathrm{C}(K^\bullet_x \xrightarrow{f_x} L^\bullet_x)$

Corollary 2.6. *Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes of diagrams. Then f is a quasi-isomorphism if and only if for every $x \in X$, $f_x : K^\bullet_x \to L^\bullet_x$ is a quasi-isomorphism.*

Proof. Let $x \in X$ and $i \in \mathbb{Z}$. Then by Lemmas 2.4 and 2.5,

$$\mathrm{H}^i(\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet))_x = \mathrm{H}^i(\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)_x) = \mathrm{H}^i(\mathrm{C}(K^\bullet_x \xrightarrow{f_x} L^\bullet_x))$$

hence $\mathrm{C}(K^\bullet \xrightarrow{f} L^\bullet)$ is acyclic if and only if $\mathrm{C}(K^\bullet_x \xrightarrow{f_x} L^\bullet_x)$ are acyclic for every $x \in X$. Using Lemma 2.2, we see that f is a quasi-isomorphism if and only if all the f_x are quasi-isomorphisms. $\square$

2.3. **Universal derived equivalence.** Recall that the derived category $D(\mathcal{B})$ of an abelian category $\mathcal{B}$ is obtained by formally inverting all the quasi-isomorphisms in $C(\mathcal{B})$. It admits a structure of a triangulated category where the distinguished triangles in $D(\mathcal{B})$ are those isomorphic to $K' \to K \to K'' \to K'[1]$ where $0 \to K' \to K \to K'' \to 0$ is a short exact sequence in $C(\mathcal{B})$.

Definition 2.7. Two posets X and Y are *universally derived equivalent* if for any abelian category $\mathcal{A}$, the derived categories $D(\mathcal{A}^X)$ and $D(\mathcal{A}^Y)$ are equivalent as triangulated categories.

Lemma 2.8. *Let X and Y be universally derived equivalent. Then X^{op} and Y^{op} are universally derived equivalent.*

Lemma 2.9. *Let X_1, Y_1 and X_2, Y_2 be two pairs of universally derived equivalent posets. Then $X_1 \times X_2$ and $Y_1 \times Y_2$ are universally derived equivalent.*

3. Formulas

Throughout this section, the poset X is fixed.

3.1. **The category $\mathcal{C}_X$.** Viewing $X \times \mathbb{Z}$ as a small category with a unique map $(x, m) \to (x', m')$ if $x \le x'$ and $m \le m'$ and no maps otherwise, we can consider the additive category $\widetilde{\mathcal{C}}_X$ whose objects are finite sequences $\{(x_i, m_i)\}_{i=1}^n$ with morphisms $\{(x_i, m_i)\}_{i=1}^n \to \{(x'_j, m'_j)\}_{j=1}^{n'}$ specified by $n' \times n$ integer matrices $(c_{ji})_{i,j}$ satisfying $c_{ji} = 0$ unless $(x_i, m_i) \le (x'_j, m'_j)$. That is, a morphism is a formal $\mathbb{Z}$-linear combination of arrows $(x_i, m_i) \to (x'_j, m'_j)$. Addition and composition of morphisms correspond to the usual addition and multiplication of matrices.

To encode the fact that squares of differentials are zero, we consider a certain quotient of $\widetilde{\mathcal{C}}_X$. Namely, let $\widetilde{\mathcal{I}}_X$ be the ideal in $\widetilde{\mathcal{C}}_X$ generated by all the morphisms $(x, m) \to (x, m+2)$ for $(x, m) \in X \times \mathbb{Z}$ and let $\mathcal{C}_X = \widetilde{\mathcal{C}}_X / \widetilde{\mathcal{I}}_X$ be the quotient. The objects of $\mathcal{C}_X$ are still sequences $\xi = \{(x_i, m_i)\}$ and the morphisms can again be written as integer matrices, albeit not uniquely as we ignore the entries c_{ji} whenever $m'_j - m_i \ge 2$.

Define a translation functor $[1] : \mathcal{C}_X \to \mathcal{C}_X$ as follows. For an object $\xi = \{(x_i, m_i)\}_{i=1}^n$, let $\xi[1] = \{(x_i, m_i + 1)\}_{i=1}^n$. For a morphism $\varphi = (c_{ji}) : \{(x_i, m_i)\} \to \{(x'_j, m'_j)\}$, let $\varphi[1]$ be the morphism $\{(x_i, m_i + 1)\} \to \{(x'_j, m'_j + 1)\}$ specified by the same matrix (c_{ji}).

Let $\mathcal{A}$ be an abelian category. From now on we shall denote a complex in $C(\mathcal{A}^X)$ by K instead of $K^\bullet$, and use Lemma 2.3 to identify $C(\mathcal{A}^X)$ with $C(\mathcal{A})^X$. Therefore we may think of K as a diagram of complexes in $C(\mathcal{A})$ and use the notations K_x, d_x, $r_{xx'}$ as in the proof of that lemma.

For two additive categories $\mathcal{B}$ and $\mathcal{B}'$, let $\mathrm{Func}(\mathcal{B}, \mathcal{B}')$ denote the category of additive functors $\mathcal{B} \to \mathcal{B}'$, with natural transformations as morphisms.

Proposition 3.1. *There exists a functor $\eta : \mathcal{C}_X \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))$ commuting with the translations.*

Proof. An object $\xi = \{(x_i, m_i)\}_{i=1}^n$ defines an additive functor F_ξ from $C(\mathcal{A})^X$ to $C(\mathcal{A})$ by sending $K \in C(\mathcal{A})^X$ and a morphism $f : K \to K'$ to

$$F_\xi(K) = \bigoplus_{i=1}^n K_{x_i}[m_i] \qquad F_\xi(f) = \bigoplus_{i=1}^n f_{x_i}[m_i] \tag{3.1}$$

where the right term is the $n \times n$ diagonal matrix whose (i, i) entry is $f_{x_i}[m_i] : K_{x_i}[m_i] \to K'_{x_i}[m_i]$.

To define η on morphisms $\xi \to \xi'$, consider first the case that $\xi = (x, m)$ and $\xi' = (x', m')$. A morphism $\varphi = (c) : (x, m) \to (x', m')$ in $\mathcal{C}_X$ is specified by an integer c, with $c = 0$ unless $(x, m) \le (x', m')$. Given $K \in C(\mathcal{A})^X$, define a morphism $\eta_\varphi(K) : K_x[m] \to K_{x'}[m']$ by

$$\eta_\varphi(K) = \begin{cases} c \cdot r_{xx'}[m] & \text{if } m' = m \text{ and } x' \geq x \\ c \cdot d_{x'}[m] r_{xx'}[m] & \text{if } m' = m+1 \text{ and } x' \geq x \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Then $\eta_c : F_\xi \to F_{\xi'}$ is a natural transformation since the diagrams

$$\begin{array}{ccccccc} K_x[m] & \xrightarrow{r_{xx'}[m]} & K_{x'}[m] & \quad & K_x[m] & \xrightarrow{d_x[m]} & K_x[m+1] \\ \downarrow{\scriptstyle f_x[m]} & & \downarrow{\scriptstyle f_{x'}[m]} & & \downarrow{\scriptstyle f_x[m]} & & \downarrow{\scriptstyle f_x[m+1]} \\ K'_x[m] & \xrightarrow[r'_{xx'}[m]]{} & K'_{x'}[m] & & K'_x[m] & \xrightarrow[d'_x[m]]{} & K'_x[m+1] \end{array} \tag{3.3}$$

commute.

Let $\varphi' = (c') : (x', m') \to (x'', m'')$ be another morphism in $\mathcal{C}_X$. Then (3.2) and the three relations $r_{xx''} = r_{x'x''} r_{xx'}$, $r_{xx'}[1] d_x = d_{x'} r_{xx'}$ and $d_x[1] d_x = 0$, imply that

$$\eta_{\varphi'\varphi}(K) = \eta_{\varphi'}(K)\eta_\varphi(K) \tag{3.4}$$

for every $K \in C(\mathcal{A})^X$.

Now for a general morphism $\varphi : \{(x_i, m_i)\}_{i=1}^n \to \{(x'_j, m'_j)\}_{j=1}^{n'}$, define morphisms $\eta_\varphi(K) : \bigoplus_{i=1}^n K_{x_i}[m_i] \to \bigoplus_{j=1}^{n'} K_{x'_j}[m'_j]$ by

$$(\eta_\varphi)_{ji} = \eta_{(c_{ji})} : K_{x_i}[m_i] \to K_{x'_j}[m'_j] \tag{3.5}$$

where $\eta_{(c_{ji})}$ is defined by (3.2) for $c_{ji} : (x_i, m_i) \to (x'_j, m'_j)$.

It follows from (3.3) by linearity that for $f : K \to K'$,

$$F_{\xi'}(f)\eta_\varphi(K) = \eta_\varphi(K')F_\xi(f) \tag{3.6}$$

so that $\eta_\varphi : F_\xi \to F_{\xi'}$ is a natural transformation. Linearity also shows that (3.4) holds for general morphisms φ, φ'.

Finally, note that by (3.1) and (3.2),

$$[1] \circ F_\xi = F_\xi \circ [1] = F_{\xi[1]} \qquad [1] \circ \eta_\varphi = \eta_\varphi \circ [1] = \eta_{\varphi[1]}$$

for any object ξ and morphism φ. □

3.2. **Formula to a point.** So far the differentials on the complexes $F_\xi(K)$ were just the direct sums $\bigoplus_{i=1}^n d_{x_i}[m_i]$. For the applications, more general differentials are needed.

Let $\varphi = (c_{ji}) : \xi \to \xi'$ be a morphism. Define $\varphi^\star : \xi \to \xi'$ by $\varphi^\star = (c^\star_{ji})$ where $c^\star_{ji} = (-1)^{m'_j - m_i} c_{ji}$.

Lemma 3.2. *Let $D : \xi \to \xi[1]$ be a morphism and assume that $D^\star[1] \cdot D = 0$ in $\mathcal{C}_X$. Then for any $K \in C(\mathcal{A})^X$, $\eta_D(K)$ is a differential on $F_\xi(K)$.*

Proof. Since $F_{\xi[1]}(K) = F_\xi(K)[1]$, the morphism D induces a map $\eta_D(K) : F_\xi(K) \to F_\xi(K)[1]$. Thinking of $\eta_D(K)$ as a potential differential, observe that

$$\eta_D(K)[1] = \eta_{-D^\star[1]}(K) \tag{3.7}$$

Indeed, each component $K_x[m+1] \to K_{x'}[m'+1]$ of $\eta_D(K)[1]$ is obtained from $K_x[m] \to K_{x'}[m']$ by a change of sign. When $m' = m$, changing the sign of a map $r_{xx'}[m]$ leads to the map $-r_{xx'}[m+1]$. When $m' = m+1$, changing the sign of $d_{x'}[m]r_{xx'}[m]$ leads to $d_{x'}[m+1]r_{xx'}[m+1]$, as the sign change is already carried out in the shift of the differential $d_{x'}[m]$. Therefore in both cases a the coefficient c of $(x,m) \to (x',m')$ changes to $-c^\star$.

Now the claim follows from

$$\eta_D(K)[1] \cdot \eta_D(K) = \eta_{-D^\star[1]}(K)\eta_D(K) = \eta_{-D^\star[1]D}(K) = 0$$

□

Definition 3.3. A morphism $\varphi = (c) : (x,m) \to (x',m')$ is a *differential* if $m' = m+1$, $x' = x$ and $c = 1$. φ is a *restriction* if $m' = m$ and $x' \geq x$.

A morphism $\varphi : \xi \to \xi'$ is a *restriction* if all its nonzero components are restrictions.

Definition 3.4. A *formula to a point* is a pair (ξ, D) where $\xi = \{(x_i, m_i)\}_{i=1}^n$ is an object of $\mathcal{C}_X$ and $D = (D_{ji})_{i,j=1}^n : \xi \to \xi[1]$ is morphism satisfying:

(1) $D^\star[1] \cdot D = 0$.
(2) $D_{ji} = 0$ for all $i > j$.
(3) D_{ii} are differentials for all $1 \leq i \leq n$.

A *morphism of formulas to a point* $\varphi : (\xi, D) \to (\xi', D')$ is a morphism $\varphi : \xi \to \xi'$ in $\mathcal{C}_X$ which is a restriction and satisfies $\varphi[1]D = D'\varphi$.

Denote by $\mathcal{F}_X$ the category of formulas to a point and their morphisms. The translation [1] of $\mathcal{C}_X$ induces a translation [1] on $\mathcal{F}_X$ by $(\xi, D)[1] = (\xi[1], D[1])$ with the same action on morphisms.

Proposition 3.5. *There exists a functor* $\eta : \mathcal{F}_X \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))$.

Proof. We actually show that the required functor is induced from the functor η of Proposition 3.1.

An object (ξ, D) defines an additive functor $F_{\xi,D} : C(\mathcal{A})^X \to C(\mathcal{A})$ by sending $K \in C(\mathcal{A})^X$ and $f : K \to K'$ to

$$F_{\xi,D}(K) = F_\xi(K) \qquad\qquad F_{\xi,D}(f) = F_\xi(f)$$

as in (3.1). By Lemma 3.2, $\eta_D(K)$ is a differential on $F_\xi(K)$.

Now observe that $F_\xi(f)[1]\eta_D(K) = \eta_D(K')F_\xi(f)$ since $\eta_D : F_\xi \to F_{\xi[1]}$ is a natural transformation. Therefore $F_\xi(f)$ is a morphism of complexes and $F_{\xi,D}$ is a functor.

Let $\varphi : (\xi, D) \to (\xi', D')$ be a morphism in $\mathcal{F}_X$. Since $\varphi : \xi \to \xi'$ in $\mathcal{C}_X$, we have a natural transformation $\eta_\varphi : F_\xi \to F_{\xi'}$. It remains to show that $\eta_\varphi(K)$ is a morphism of complexes. But the commutativity with the differentials $\eta_D(K)$ and $\eta_{D'}(K)$ follows from $\varphi[1]D = D'\varphi$ and the functoriality of η. □

Example 3.6 (Zero dimensional chain)**.** Let $x \in X$ and consider $\xi = \{(x,0)\}$ with $D = (1)$. The functor $F_{(x,0),(1)}$ sends K to the stalk K_x and $f : K \to K'$ to f_x.

Example 3.7 (One dimensional chain)**.** Let $x < y$ in X and consider $\xi = \{(x,1),(y,0)\}$ with the map $D = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) : \xi \to \xi[1]$. Then for $K \in C(\mathcal{A})^X$ and $f : K \to K'$,

$$F_{\xi,D}(K) = K_x[1] \oplus K_y \qquad\qquad F_{\xi,D}(f) = \begin{pmatrix} f_x[1] & 0 \\ 0 & f_y \end{pmatrix}$$

with the differential

$$\eta_D(K) = \begin{pmatrix} d_x[1] & 0 \\ r_{xy}[1] & d_y \end{pmatrix} : K_x[1] \oplus K_y \to K_x[2] \oplus K_y[1]$$

Since for any object K, $F_{\xi,D}(K) = \mathrm{C}(K_x \xrightarrow{r_{xy}} K_y)$ as complexes, we see that for any $x < y$, the cone $\mathrm{C}(K_x \xrightarrow{r_{xy}} K_y)$ defines a functor $C(\mathcal{A})^X \to C(\mathcal{A})$.

Lemma 3.8. *There exists a natural isomorphism* $\varepsilon : [1] \circ \eta \xrightarrow{\simeq} \eta \circ [1]$.

Proof. We first remark that for an object $(\xi, D) \in \mathcal{F}_X$, a morphism φ and $K \in C(\mathcal{A})^X$, $F_{\xi[1],D[1]}(K) = F_{\xi,D}(K[1])$ and $\eta_{\varphi[1]}(K) = \eta_\varphi(K[1])$ so that $(\eta \circ [1])(\xi, D)$ can be viewed as first applying the shift on $C(\mathcal{A})^X$ and then applying $F_{\xi,D}$.

We will construct natural isomorphisms of functors $\varepsilon_{\xi,D} : [1] \circ F_{\xi,D} \xrightarrow{\simeq} F_{\xi,D} \circ [1]$ such that the diagrams

$$\begin{array}{ccc} F_{\xi,D}(K)[1] & \xrightarrow{\varepsilon_{\xi,D}} & F_{\xi[1],D[1]}(K) \\ \downarrow{\scriptstyle [1]\circ\eta_\varphi} & & \downarrow{\scriptstyle \eta_{\varphi[1]}} \\ F_{\xi',D'}(K)[1] & \xrightarrow[\varepsilon_{\xi',D'}]{} & F_{\xi'[1],D'[1]}(K) \end{array} \tag{3.8}$$

commute for all $K \in C(\mathcal{A})^X$.

By (3.7), $[1] \circ F_{\xi,D} = F_{\xi[1],-D^\star[1]}$. Write $\xi = \{(x_i, m_i)\}_{i=1}^n$, $D = (D_{ji})_{i,j=1}^n$, and let $I_\xi : \xi \to \xi$ be the morphism defined by the diagonal matrix whose (i,i) entry is $(-1)^{m_i}$. By definition, $D^\star_{ji} = (-1)^{m_j+1-m_i} D_{ji}$, or equivalently $(-1)^{m_j} D_{ji} = -D^\star_{ji}(-1)^{m_i}$ for all i,j, hence $I_\xi[1]D = -D^\star I_\xi$. Therefore $I_\xi[1] : (\xi[1], D[1]) \to (\xi[1], -D^\star[1])$ is an isomorphism in $\mathcal{F}_X$, so we define $\varepsilon_{\xi,D} = \eta_{I_\xi[1]}$.

For the commutativity of (3.8), first observe that $[1] \circ \eta_\varphi = \eta_\varphi \circ [1] = \eta_{\varphi[1]}$. Now use the fact that $I_{\xi'}\varphi = \varphi I_\xi$ for any restriction $\varphi : \xi \to \xi'$. □

In the next few lemmas, we fix a formula to a point (ξ, D).

Lemma 3.9. *$F_{\xi,D}$ maps short exact sequences to short exact sequences.*

Proof. Write $\xi = \{(x_i, m_i)\}_{i=1}^n$ and let $0 \to K' \xrightarrow{f'} K \xrightarrow{f''} K'' \to 0$ be a short exact sequence. Then $0 \to K'_x \xrightarrow{f'_x} K_x \xrightarrow{f''_x} K''_x \to 0$ is exact for any $x \in X$, hence

$$0 \to \bigoplus_{i=1}^n K'_{x_i}[m_i] \xrightarrow{\bigoplus_{i=1}^n f'_{x_i}[m_i]} \bigoplus_{i=1}^n K_{x_i}[m_i] \xrightarrow{\bigoplus_{i=1}^n f''_{x_i}[m_i]} \bigoplus_{i=1}^n K''_{x_i}[m_i] \to 0$$

is exact. □

By composing with the equivalence $\Phi : C(\mathcal{A}^X) \to C(\mathcal{A})^X$, we may view $F_{\xi,D}$ as a functor $C(\mathcal{A}^X) \to C(\mathcal{A})$ between two categories of complexes.

Lemma 3.10. *$F_{\xi,D}$ maps quasi-isomorphisms to quasi-isomorphisms.*

Proof. Write $\xi = \{(x_i, m_i)\}_{i=1}^n$. We prove the claim by induction on n. When $n = 1$, we have $\xi = (x, m)$, $F_{\xi,D}(K) = K_x[m]$ and $F_{\xi,D}(f) = f_x[m]$, so that the claim follows from Corollary 2.6.

Assume now that $n > 1$, and let $\xi' = \{(x_i, m_i)\}_{i=1}^{n-1}$ and $D' = (D_{ji})_{i,j=1}^{n-1}$ be the corresponding restricted matrix. By the assumption that $D = (D_{ji})$ is lower triangular with ones on the main diagonal, we have that the canonical embedding $\iota_K : K_{x_n}[m_n] \to \bigoplus_{i=1}^{n} K_{x_i}[m_i]$ and the projection $\pi_K : \bigoplus_{i=1}^{n} K_{x_i}[m_i] \to \bigoplus_{i=1}^{n-1} K_{x_i}[m_i]$ commute with the differentials, hence there exists a functorial short exact sequence
(3.9)

$$0 \to (K_{x_n}[m_n], d_{x_n}[m_n]) \to (F_{\xi,D}(K), \eta_D(K)) \to (F_{\xi',D'}(K), \eta_{D'}(K)) \to 0$$

Let $f : K \to K'$ be a morphism. The functoriality of (3.9) gives rise to the following diagram of long exact sequences in cohomology,

$$\begin{array}{ccccccc}
\longrightarrow & \mathrm{H}^i(F_{\xi',D'}(K)) & \longrightarrow & \mathrm{H}^i(K_{x_n}[m_n]) & \longrightarrow & \mathrm{H}^i(F_{\xi,D}(K)) = \\
& \Big\downarrow {\scriptstyle \mathrm{H}^i(F_{\xi',D'}(f))} & & \Big\downarrow {\scriptstyle \mathrm{H}^i(f_{x_n}[m_n])} & & \Big\downarrow {\scriptstyle \mathrm{H}^i(F_{\xi,D}(f))} \\
\longrightarrow & \mathrm{H}^i(F_{\xi',D'}(K')) & \longrightarrow & \mathrm{H}^i(K'_{x_n}[m_n]) & \longrightarrow & \mathrm{H}^i(F_{\xi,D}(K')) =
\end{array}$$

$$\begin{array}{ccccccc}
= \mathrm{H}^i(F_{\xi,D}(K)) & \longrightarrow & \mathrm{H}^{i+1}(F_{\xi',D'}(K)) & \longrightarrow & \mathrm{H}^{i+1}(K_{x_n}[m_n]) & \longrightarrow \\
\Big\downarrow {\scriptstyle \mathrm{H}^i(F_{\xi,D}(f))} & & \Big\downarrow {\scriptstyle \mathrm{H}^{i+1}(F_{\xi',D'}(f))} & & \Big\downarrow {\scriptstyle \mathrm{H}^{i+1}(f_{x_n}[m_n])} \\
= \mathrm{H}^i(F_{\xi,D}(K')) & \longrightarrow & \mathrm{H}^{i+1}(F_{\xi',D'}(K')) & \longrightarrow & \mathrm{H}^{i+1}(K'_{x_n}[m_n]) & \longrightarrow
\end{array}$$

Now assume that $f : K \to K'$ is a quasi-isomorphism. By the induction hypothesis, $f_{x_n}[m_n] : K_{x_n}[m_n] \to K'_{x_n}[m_n]$ and $F_{\xi',D'}(f) : F_{\xi',D'}(K) \to F_{\xi',D'}(K')$ are quasi-isomorphisms, hence by the Five Lemma, $F_{\xi,D}(f)$ is also a quasi-isomorphism. □

Corollary 3.11. *Let (ξ, D) be a formula to a point. Then $F_{\xi,D}$ induces a triangulated functor $\widetilde{F}_{\xi,D} : D(\mathcal{A}^X) \to D(\mathcal{A})$.*

3.3. General formulas.

Definition 3.12. Let Y be a poset. A *formula from X to Y* is a diagram over Y with values in $\mathcal{F}_X$.

Proposition 3.13. *There exists a functor $\eta : \mathcal{F}_X^Y \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A})^Y)$.*

Proof. Let $\eta : \mathcal{F}_X \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))$ be the functor of Proposition 3.5. Then

$$\eta^Y : \mathcal{F}_X^Y \to \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A}))^Y \simeq \mathrm{Func}(C(\mathcal{A})^X, C(\mathcal{A})^Y)$$

is the required functor. □

Let $\boldsymbol{\xi} \in \mathcal{F}_X$ be a formula.

Lemma 3.14. *$F_{\boldsymbol{\xi}}$ maps short exact sequences to short exact sequences.*

Proof. It is enough to consider each component of $F_{\boldsymbol{\xi}}$ separately. The claim now follows from Lemma 3.9. □

By composing from the left with the equivalence $\Phi : C(\mathcal{A}^X) \to C(\mathcal{A})^X$ and from the right with $\Phi^{-1} : C(\mathcal{A})^Y \to C(\mathcal{A}^Y)$ we may view $F_{\boldsymbol{\xi}}$ as a functor $C(\mathcal{A}^X) \to C(\mathcal{A}^Y)$ between two categories of complexes.

Lemma 3.15. *$F_{\boldsymbol{\xi}}$ maps quasi-isomorphisms to quasi-isomorphisms.*

Proof. Let $f : K \to K'$ be a quasi-isomorphism. By Corollary 2.6, it is enough to show that each component of $F_{\boldsymbol{\xi}}(f)$ is a quasi-isomorphism in $C(\mathcal{A})$. But this follows from Lemma 3.10. □

Corollary 3.16. *Let $\boldsymbol{\xi}$ be a formula. Then $F_{\boldsymbol{\xi}}$ induces a triangulated functor $\widetilde{F}_{\boldsymbol{\xi}} : D(\mathcal{A}^X) \to D(\mathcal{A}^Y)$.*

4. Applications of formulas

4.1. **The chain with two elements.** As a first application we consider the case where the poset X is a chain of two elements

$$\bullet_1 \longrightarrow \bullet_2$$

We focus on this simple case as the fundamental underlying principle of Theorem 1.1 can already be effectively demonstrated in that case.

Let (ξ_1, D_1), (ξ_2, D_2) and (ξ_{12}, D_{12}) be the following three formulas to a point in $\mathcal{F}_{1\to 2}$.

$$\begin{aligned} &\xi_1 = (1,1), D_1 = (1) \qquad \xi_{12} = ((1,1),(2,0)), D_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &\xi_2 = (2,0), D_2 = (1) \end{aligned} \tag{4.1}$$

Let $\mathcal{A}$ be an abelian category and $K = K_1 \xrightarrow{r_{12}} K_2$ be an object of $C(\mathcal{A}^{1\to 2}) \simeq C(\mathcal{A})^{1\to 2}$. In the more familiar notation,

$$F_{\xi_1,D_1}(K) = K_1[1] \quad F_{\xi_2,D_2}(K) = K_2 \quad F_{\xi_{12},D_{12}}(K) = \mathrm{C}(K_1 \xrightarrow{r_{12}} K_2)$$

see Examples 3.6 and 3.7.

The morphisms

$$\varphi_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} : \xi_{12} \to \xi_1 \qquad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \xi_2 \to \xi_{12}$$

are restrictions that satisfy $\varphi_1 D_{12} = D_1 \varphi_1$ and $\varphi_2 D_2 = D_{12} \varphi_2$, hence

$$\boldsymbol{\xi}^- = (\xi_{12}, D_{12}) \xrightarrow{\varphi_1} (\xi_1, D_1) \qquad \boldsymbol{\xi}^+ = (\xi_2, D_2) \xrightarrow{\varphi_2} (\xi_{12}, D_{12})$$

are diagrams over $1 \to 2$ with values in $\mathcal{F}_{1\to 2}$, thus they define functors $R^-, R^+ : C(\mathcal{A}^{1\to 2}) \to C(\mathcal{A}^{1\to 2})$ inducing triangulated functors $\widetilde{R}^-, \widetilde{R}^+ : D(\mathcal{A}^{1\to 2}) \to D(\mathcal{A}^{1\to 2})$. Their values on objects $K \in C(\mathcal{A}^{1\to 2})$ are

$$\begin{aligned} R^-(K) &= \mathrm{C}(K_1 \xrightarrow{r_{12}} K_2) \xrightarrow{(r_{11}[1]\ 0)} K_1[1] \\ R^+(K) &= K_2 \xrightarrow{\left(\begin{smallmatrix} 0 \\ r_{22} \end{smallmatrix}\right)} \mathrm{C}(K_1 \xrightarrow{r_{12}} K_2) \end{aligned} \tag{4.2}$$

Proposition 4.1. *There are natural transformations*

$$R^+ \circ R^- \xrightarrow{\varepsilon^{+-}} [1] \xrightarrow{\varepsilon^{-+}} R^- \circ R^+$$

such that $\varepsilon^{+-}(K)$, $\varepsilon^{-+}(K)$ are quasi-isomorphisms for all $K \in C(\mathcal{A}^{1\to 2})$.

Proof. The functors $R^+ \circ R^-$ and $R^- \circ R^+$ correspond to the compositions $\boldsymbol{\xi}^{+-} = \boldsymbol{\xi}^+ \circ (\boldsymbol{\xi}_1^- \to \boldsymbol{\xi}_2^-)$ and $\boldsymbol{\xi}^{-+} = \boldsymbol{\xi}^- \circ (\boldsymbol{\xi}_1^+ \to \boldsymbol{\xi}_2^+)$, given by

$$\boldsymbol{\xi}^{+-} = (\xi_1, D_1) \xrightarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} (\xi_{121}, D_{121})$$
$$\boldsymbol{\xi}^{-+} = (\xi_{212}, D_{212}) \xrightarrow{(1\ 0\ 0)} (\xi_2[1], D_2[1])$$

where

$$(\xi_{121}, D_{121}) = \left(((1,2),(2,1),(1,1)), \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right) \tag{4.3}$$
$$(\xi_{212}, D_{212}) = \left(((2,1),(1,1),(2,0)), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right)$$

and the translation [1] corresponds to the diagram

$$\boldsymbol{\nu} = (\xi_1, D_1) \xrightarrow{(1)} (\xi_2[1], D_2[1])$$

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the morphisms

(4.4)

$$\alpha_1 : (\xi_1, D_1) \xrightarrow{\left(\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right)} (\xi_{212}, D_{212}) \qquad \beta_1 : (\xi_{212}, D_{212}) \xrightarrow{(0\ -1\ 0)} (\xi_1, D_1)$$
$$\alpha_2 : (\xi_2[1], D_2[1]) \xrightarrow{\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)} (\xi_{121}, D_{121}) \quad \beta_2 : (\xi_{121}, D_{121}) \xrightarrow{(0\ 1\ 1)} (\xi_2[1], D_2[1])$$

The following diagram in $\mathcal{F}_{1\to 2}$

$$\begin{array}{ccccc}
(\xi_1, D_1) & \xrightarrow{(1)} & (\xi_1, D_1) & \xrightarrow{\alpha_1} & (\xi_{212}, D_{212}) \\
\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)\downarrow & & \downarrow(1) & & \downarrow(1\ 0\ 0) \\
(\xi_{121}, D_{121}) & \xrightarrow[\beta_2]{} & (\xi_2[1], D_2[1]) & \xrightarrow{(1)} & (\xi_2[1], D_2[1])
\end{array}$$

is commutative, hence the horizontal arrows induce morphisms of formulas $\boldsymbol{\xi}^{+-} \to \boldsymbol{\nu}$ and $\boldsymbol{\nu} \to \boldsymbol{\xi}^{-+}$, inducing natural transformations $\varepsilon^{+-} : R^+R^- \to [1]$ and $\varepsilon^{-+} : [1] \to R^-R^+$.

We prove that $\varepsilon^{+-}(K)$ and $\varepsilon^{-+}(K)$ are quasi-isomorphisms for all K by showing that each component is a quasi-isomorphism (see Corollary 2.6). Indeed, let $h_1 : \xi_{212} \to \xi_{212}[-1]$ and $h_2 : \xi_{121} \to \xi_{121}[-1]$ be the maps

$$h_1 = h_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$\beta_1\alpha_1 = (1) \qquad \alpha_1\beta_1 + (h_1[1]D_{212} + D_{212}^{\star}[-1]h_1) = I_3 \tag{4.5}$$
$$\beta_2\alpha_2 = (1) \qquad \alpha_2\beta_2 + (h_2[1]D_{121} + D_{121}^{\star}[-1]h_2) = I_3$$

where I_3 is the 3×3 identity matrix, hence $\beta_1\alpha_1$ and $\beta_2\alpha_2$ induce the identities and $\alpha_1\beta_1$, $\alpha_2\beta_2$ induce morphisms $\eta_{\alpha_1\beta_1}(K)$ and $\eta_{\alpha_2\beta_2}(K)$ homotopic to the identities. Therefore $\eta_{\alpha_1}(K)$, $\eta_{\alpha_2}(K)$, $\eta_{\beta_1}(K)$ and $\eta_{\beta_2}(K)$ are quasi-isomorphisms. □

Proposition 4.2. *There are natural transformations*

$$R^+ \circ R^+ \xrightarrow{\varepsilon^{++}} R^- \qquad\qquad R^+ \circ [1] \xrightarrow{\varepsilon^{--}} R^- \circ R^-$$

such that $\varepsilon^{++}(K)$, $\varepsilon^{--}(K)$ are quasi-isomorphisms for all $K \in C(\mathcal{A}^{1\to 2})$.

Proof. The functors $R^+ \circ R^+$ and $R^- \circ R^-$ correspond to the compositions $\boldsymbol{\xi}^{++} = \boldsymbol{\xi}^+ \circ (\boldsymbol{\xi}_1^+ \to \boldsymbol{\xi}_2^+)$ and $\boldsymbol{\xi}^{--} = \boldsymbol{\xi}^- \circ (\boldsymbol{\xi}_1^- \to \boldsymbol{\xi}_2^-)$, given by

$$\boldsymbol{\xi}^{++} = (\xi_{12}, D_{12}) \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} (\xi_{212}, D_{212})$$
$$\boldsymbol{\xi}^{--} = (\xi_{121}, D_{121}) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)} (\xi_{12}[1], -D_{12}^{\star}[1])$$

where (ξ_{121}, D_{121}) and (ξ_{212}, D_{212}) are as in (4.3). The commutative diagrams

$$\begin{array}{ccc} (\xi_{12}, D_{12}) & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} & (\xi_{12}, D_{12}) \\ {\scriptstyle\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\big\downarrow & & \big\downarrow{\scriptstyle (1\ 0)} \\ (\xi_{212}, D_{212}) & \xrightarrow{-\beta_1} & (\xi_1[1], D_1[1]) \end{array} \qquad \begin{array}{ccc} (\xi_2[1], D_2[1]) & \xrightarrow{\alpha_2} & (\xi_{121}, D_{121}) \\ {\scriptstyle\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}\big\downarrow & & \big\downarrow{\scriptstyle\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)} \\ (\xi_{12}[1], D_{12}[1]) & \xrightarrow{\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} & (\xi_{12}[1], -D_{12}^{\star}[1]) \end{array}$$

where α_2, β_1 are as in (4.4), define morphisms of formulas $\boldsymbol{\xi}^{++} \to \boldsymbol{\xi}^-$ and $\boldsymbol{\xi}^+[1] \to \boldsymbol{\xi}^{--}$, hence natural transformations $\varepsilon^{++} : R^+R^+ \to R^-$ and $\varepsilon^{--} : R^+[1] \to R^-R^-$. Using the homotopies (4.5), one proves that $\varepsilon^{++}(K)$ and $\varepsilon^{--}(K)$ are quasi-isomorphisms for all K in the same way as before. □

Corollary 4.3. *For any abelian category $\mathcal{A}$, the functors $\widetilde{R}^+$ and $\widetilde{R}^-$ are auto-equivalences of $D(\mathcal{A})^{1\to 2}$ satisfying*

$$\widetilde{R}^+\widetilde{R}^- \simeq [1] \simeq \widetilde{R}^-\widetilde{R}^+ \qquad (\widetilde{R}^+)^2 \simeq \widetilde{R}^- \qquad (\widetilde{R}^-)^2 \simeq \widetilde{R}^+ \circ [1]$$

hence $(\widetilde{R}^+)^3 \simeq [1]$.

4.2. **Proof of Theorem 1.1.** Let X and Y be two posets satisfying the assumptions (1.1) and (1.2), and let $\leq_+$, $\leq_-$ be the partial orders on $X \sqcup Y$ as defined by (1.4). We will prove the universal derived equivalence of $\leq_+$ and $\leq_-$ by defining two formulas $\boldsymbol{\xi}^+$, $\boldsymbol{\xi}^-$ that will induce, for any abelian category $\mathcal{A}$, functors

$$R^+ = F_{\boldsymbol{\xi}^+} : C(\mathcal{A})^{\leq_+} \to C(\mathcal{A})^{\leq_-} \qquad R^- = F_{\boldsymbol{\xi}^-} : C(\mathcal{A})^{\leq_-} \to C(\mathcal{A})^{\leq_+}$$

and

$$\widetilde{R}^+ = \widetilde{F}_{\boldsymbol{\xi}^+} : D(\mathcal{A}^{\leq_+}) \to D(\mathcal{A}^{\leq_-}) \qquad \widetilde{R}^- = \widetilde{F}_{\boldsymbol{\xi}^-} : D(\mathcal{A}^{\leq_-}) \to D(\mathcal{A}^{\leq_+})$$

such that $\widetilde{R}^+\widetilde{R}^- \simeq [1]$ and $\widetilde{R}^-\widetilde{R}^+ \simeq [1]$.

4.2.1. *Definition of the formulas to points.* For $x \in X$ and $y \in Y$, let

$$\xi_x = \big((x,0),(1)\big) \qquad \xi_y = \big((y,0),(1)\big) \qquad \xi_{Y_x} = ((y,0)_{y \in Y_x}, I)$$

where I is the identity matrix. We consider ξ_x, ξ_y and ξ_{Y_x} as formulas either in $\mathcal{F}_{\leq_+}$ or in $\mathcal{F}_{\leq_-}$, as appropriate. If $y \in Y$, define

$$\boldsymbol{\xi}_y^+ = \xi_y \in \mathcal{F}_{\leq_+} \qquad \boldsymbol{\xi}_y^- = \xi_y \in \mathcal{F}_{\leq_-}$$

as in Example 3.6. If $x \in X$, let

$$\xi_{x,Y_x} = \big(\xi_x \xrightarrow{\left(\begin{smallmatrix}1\\1\\ \vdots\\ 1\end{smallmatrix}\right)} \xi_{Y_x}\big) \in \mathcal{F}_{\leq_+}^{1\to 2} \qquad \xi_{Y_x,x} = \big(\xi_{Y_x} \xrightarrow{(1\ 1\ \dots\ 1)} \xi_x\big) \in \mathcal{F}_{\leq_-}^{1\to 2}$$

be formulas to $1 \to 2$ and define

$$\boldsymbol{\xi}_x^+ = \xi_{12} \circ \xi_{x,Y_x} \qquad \boldsymbol{\xi}_x^- = \xi_{12} \circ \xi_{Y_x,x}$$

as compositions with the formula ξ_{12} defined in (4.1).

In explicit terms, let $K \in C(\mathcal{A})^{\leq_+}$, $L \in C(\mathcal{A})^{\leq_-}$, and denote by $\{r_{xy}\}$ the restriction maps in K and by $\{s_{yx}\}$ the restriction maps in L. For $x \in X$ and $y \in Y_x$, let $\iota_y : K_y \to \bigoplus_{y_x \in Y_x} K_{y_x}$ and $\pi_y : \bigoplus_{y_x \in Y_x} L_{y_x} \to L_y$ be the canonical inclusions and projections. Then

$$R^+(K)_x = \mathrm{C}(K_x \xrightarrow{\sum_{y \in Y_x} \iota_y r_{xy}} \bigoplus_{y \in Y_x} K_y) \qquad R^+(K)_y = K_y$$

$$R^-(L)_x = \mathrm{C}(\bigoplus_{y \in Y_x} L_y \xrightarrow{\sum_{y \in Y_x} s_{yx} \pi_y} L_x) \qquad R^-(L)_y = L_y[1]$$

for $x \in X$, $y \in Y$.

4.2.2. *Definition of the restriction maps.* We shall denote by ρ^+ the restriction maps between the formulas in R^+ and by ρ^- the maps between those in R^-. We consider several cases, and use the explicit notation.

For $y \leq y'$, define

$$\rho^+_{yy'}(K) = r_{yy'} : K_y \to K_{y'} \qquad \rho^-_{yy'}(L) = s_{yy'}[1] : L_y[1] \to L_{y'}[1]$$

For $x \leq x'$, we use the isomorphism $\varphi_{x,x'} : Y_x \to Y_{x'}$ and the property that $y \leq \varphi_{x,x'}(y)$ for all $y \in Y_x$ to define the diagonal maps

$$\rho^+_{xx'}(K) = r_{xx'}[1] \oplus (\bigoplus_{y \in Y_x} r_{y,\varphi_{xx'}(y)}) : R^+(K)_x \to R^+(K)_{x'}$$

$$\rho^-_{xx'}(L) = (\bigoplus_{y \in Y_x} s_{y,\varphi_{xx'}(y)}[1]) \oplus s_{xx'} : R^-(L)_x \to R^-(L)_{x'}$$

If $y_x \in Y_x$, then by (1.4), $y_x \leq_- x$, $x \leq_+ y_x$, and we define

$$\rho^+_{y_x x}(K) = K_{y_x} \xrightarrow{\left(\begin{smallmatrix}0\\ \iota_{y_x}\end{smallmatrix}\right)} \mathrm{C}(K_x \to \bigoplus_{y \in Y_x} K_y)$$

$$\rho^-_{x y_x}(L) = \mathrm{C}(\bigoplus_{y \in Y_x} L_y \to K_x) \xrightarrow{(\pi_{y_x}[1]\ 0)} L_{y_x}[1]$$

Finally, if $y \leq_- x$, by (1.1) there exists a *unique* $y_x \in Y_x$ such that $y \leq y_x$ and we set $\rho^+_{yx}(K) = \rho^+_{y_x x}(K)\rho^+_{yy_x}(K)$. Similarly, if $x \leq_+ y$, there exists a unique $y_x \in Y_x$ with $y_x \leq y$, and we set $\rho^-_{xy}(L) = \rho^-_{y_x y}(L)\rho^-_{xy_x}(L)$.

4.2.3. *Verification of commutativity.* Again there are several cases to consider. First, when $y \leq y' \leq y''$, $\rho^+_{yy''} = \rho^+_{y'y''}\rho^+_{yy'}$ follows from the commutativity of the restrictions $r_{yy''} = r_{y'y''}r_{yy'}$, and similarly for ρ^-.

Let $x \leq x' \leq x''$. Since $\varphi_{xx'} : Y_x \to Y_{x'}$ is an isomorphism and $\varphi_{xx''} = \varphi_{x'x''}\varphi_{xx'}$, we can write

$$\begin{aligned}\rho^+_{x'x''}(K) &= r_{x'x''}[1] \oplus \bigoplus_{y' \in Y_{x'}} r_{y',\varphi_{x'x''}(y')} = r_{x'x''}[1] \oplus \bigoplus_{y \in Y_x} r_{\varphi_{xx'}(y),\varphi_{x'x''}\varphi_{xx'}(y)} \\ &= r_{x'x''}[1] \oplus \bigoplus_{y \in Y_x} r_{\varphi_{xx'}(y),\varphi_{xx''}(y)}\end{aligned}$$

Now $\rho^+_{xx''} = \rho^+_{x'x''}\rho^+_{xx'}$ follows from the commutativity of the restrictions $r_{xx''} = r_{x'x''}r_{xx'}$ and $r_{y,\varphi_{xx''}(y)} = r_{\varphi_{xx'}(y),\varphi_{xx''}(y)}r_{y,\varphi_{xx'}(y)}$. The proof for ρ^- is similar.

If $y' \leq y \leq_- x$, let $y_x, y'_x \in Y_x$ be the elements satisfying $y \leq y_x$, $y' \leq y'_x$. Then $y'_x = y_x$ by uniqueness, since $y' \leq y_x$. Hence

$$\rho^+_{y'x} = \rho^+_{y_x x}\rho^+_{y'y_x} = \rho^+_{y_x x}\rho^+_{yy_x}\rho^+_{y'y} = \rho^+_{yx}\rho^+_{y'y}$$

The proof for ρ^- in the case $x \leq_+ y \leq y'$ is similar.

If $y_x \leq_- x \leq x'$ where $y_x \in Y_x$, then $y_{x'} = \varphi_{xx'}(y_x)$ is the unique element $y_{x'} \in Y_{x'}$ with $y_x \leq y_{x'}$, and

$$\rho^+_{y_x x'} = \rho^+_{\varphi_{xx'}(y_x),x'}\rho^+_{y_x,\varphi_{xx'}(y_x)} = \rho^+_{xx'}\rho^+_{y_x x}$$

by the commutativity of the diagram

$$\begin{array}{ccc} K_{y_x} & \xrightarrow{\rho^+_{y_x,x}} & \mathrm{C}(K_x \to \bigoplus_{y \in Y_x} K_y) \\ {\scriptstyle r_{y_x,\varphi_{xx'}(y_x)}}\downarrow & & \downarrow{\scriptstyle \rho^+_{xx'} = r_{xx'}[1] \oplus \bigoplus r_{y,\varphi_{xx'}(y)}} \\ K_{\varphi_{x,x'}(y_x)} & \xrightarrow{\rho^+_{\varphi_{xx'}(y_x),x'}} & \mathrm{C}(K_{x'} \to \bigoplus_{y' \in Y_{x'}} K_{y'}) \end{array}$$

Now if $y \leq_- x \leq x'$, let $y_x \in Y_x$ be the element with $y \leq y_x$. Then $y \leq y_x \leq_- x \leq x'$ and commutativity follows from the previous two cases:

$$\rho^+_{yx'} = \rho^+_{y_x x'}\rho^+_{yy_x} = \rho^+_{xx'}\rho^+_{y_x x}\rho^+_{yy_x} = \rho^+_{xx'}\rho^+_{yx}$$

The proof for ρ^- in the cases $x' \leq x \leq_+ y_x$ and $x' \leq x \leq_+ y$ is similar. Here we also use fact that $\varphi_{x'x}$ is an isomorphism to pick $y_{x'} = \varphi^{-1}_{x'x}(y_x)$ as the unique element $y_{x'} \in Y_{x'}$ with $y_{x'} \leq y_x$.

4.2.4. *Construction of the natural transformations $R^+R^- \to [1] \to R^-R^+$.* Observe that

$$\begin{aligned} (\boldsymbol{\xi}^+\boldsymbol{\xi}^-)_y &= \xi_y[1] & (\boldsymbol{\xi}^-\boldsymbol{\xi}^+)_y &= \xi_y[1] \\ (\boldsymbol{\xi}^+\boldsymbol{\xi}^-)_x &= \xi_{121} \circ \xi_{Y_x,x} & (\boldsymbol{\xi}^-\boldsymbol{\xi}^+)_x &= \xi_{212} \circ \xi_{x,Y_x} \end{aligned}$$

where ξ_{121} and ξ_{212} are the formulas defined in (4.3).

Let $\boldsymbol{\nu}$ be the formula inducing the translation and define $\varepsilon^{+-} : \boldsymbol{\xi}^+\boldsymbol{\xi}^- \to \boldsymbol{\nu}$, $\varepsilon^{-+} : \boldsymbol{\nu} \to \boldsymbol{\xi}^-\boldsymbol{\xi}^+$ by

$$\varepsilon_y^{+-} : \xi_y[1] \xrightarrow{(1)} \xi_y[1]$$
$$\varepsilon_x^{+-} : \xi_{121} \circ \xi_{Y_x,x} \xrightarrow{\beta_2 \circ \xi_{Y_x,x}} \xi_2[1] \circ \xi_{Y_x,x} = \xi_x[1]$$
$$\varepsilon_y^{-+} : \xi_y[1] \xrightarrow{(1)} \xi_y[1]$$
$$\varepsilon_x^{-+} : \xi_x[1] = \xi_1 \circ \xi_{x,Y_x} \xrightarrow{\alpha_1 \circ \xi_{x,Y_x}} \xi_{212} \circ \xi_{x,Y_x,x}$$

where ξ_1 and ξ_2 are as in (4.1) and α_1 and β_2 are as in Proposition 4.1. The proof of that proposition also shows that ε^{+-} and ε^{-+} are morphisms of formulas and induce natural transformations between functors, which are quasi-isomorphisms.

4.3. **Proof of Corollary 1.6.** Let X and Z be posets, and let $Y = \mathbf{1} \oplus Z$. Denote by $1 \in Y$ the unique minimal element and consider the map $f : X \to Y$ defined by $f(x) = 1$ for all $x \in X$. Then

$$(X \sqcup Y, \leq_+^f) \simeq X \oplus \mathbf{1} \oplus Z \qquad (X \sqcup Y, \leq_-^f) \simeq \mathbf{1} \oplus (X + Z)$$

hence by Corollary 1.3, $X \oplus \mathbf{1} \oplus Z$ and $\mathbf{1} \oplus (X + Z)$ are universally derived equivalent.

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