ORTHOGONAL ROOT NUMBERS AND THE REFINED FORMAL DEGREE CONJECTURE

Let k be a local field. If k is nonarchimedean, let p be its residue characteristic and let q be the cardinality of the residue field of k. Let $\underline{G}$ be a reductive k-group, let $G = \underline{G}(k)$, and let $\underline{A}_G$ the split component of the center of $\underline{G}$. Let Γ_k be the Galois group, let W_k be the Weil group, let $\mathrm{WD}_k \stackrel{\text{def}}{=} W_k \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group, and let

$$L_k \stackrel{\text{def}}{=} \begin{cases} \mathrm{WD}_k & \text{if } k \text{ is nonarchimedean} \\ W_k & \text{if } k \text{ is archimedean} \end{cases}$$

be the Langlands group. For all local (that is, L-, ε-, and γ-) factors, we use the additive character of k for which $\mathcal{O}_k$ is the largest fractional ideal of k in the kernel and use the Haar measure on k that assigns volume one to $\mathcal{O}_k$.

Raphaël's talk introduced us to the following conjecture of Hiraga, Ichino, and Ikeda.

Conjecture 1 ([HII08b, Conjecture 1.4])**.** *Let π be a(n essentially) discrete series representation of G with extended parameter (φ, ρ). Then*

$$d(\pi, \nu_G) = \frac{\dim \rho}{|S_\varphi|} \cdot |\gamma(0, \varphi, \mathrm{Ad}_G)|.$$

Here $d(\pi)$ is the formal degree of π,

$$\gamma(s, \varphi, \mathrm{Ad}_G) \stackrel{\text{def}}{=} \varepsilon(s, \mathrm{Ad}_G \circ \varphi) \cdot \frac{L(1-s, \mathrm{Ad}_G \circ \varphi)}{L(s, \mathrm{Ad}_G \circ \varphi)},$$

ν_G is the measure on G defined in Raphaël's talk using a volume form on the Chevalley model of $\underline{G}$, and Ad_G is the adjoint representation of ${}^L G$ on $\widehat{\mathfrak{g}}/\hat{\mathfrak{z}}^\Gamma$. The precise definitions of ρ and S_φ are not so important to us; let me say only that S_φ is built from the centralizer of φ and ρ is a finite-dimensional irreducible representation of a group similar to S_φ.

In this talk I'll explain some of my work on understanding the conjecture more fully.

1. Refined formal degree

In this section, we assume k is nonarchimedean. Gross and Reeder [GR10, Section 7] refined Result 1 to remove the absolute value bars and interpret the sign of the γ-factor. This sign is a quotient of root numbers. Recall that the root number of a representation $\varphi\colon W_k \to \mathrm{GL}(V)$ is the number $\omega(\varphi)$ defined by the formula

$$\varepsilon(s, \varphi) = \omega(\varphi) q^{\mathrm{cond}(\varphi)(1/2-s)}$$

where $\mathrm{cond}(\varphi)$ is the Artin conductor. The root number has modulus one, and even better, when φ is self-dual, it is a fourth root of unity because

$$\omega(\varphi)^2 = (\det \varphi)(-1).$$

(Use Artin reciprocity to make sense of the righthand side.) Deligne has given an interpretation of the sign in terms of Stiefel-Whitney classes. His theorem has several variations,

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the simplest of which says that when $\det\varphi = 1$, the root number $\omega(\varphi)$ is $+1$ if and only if φ lifts to the spin double-cover $\mathrm{Spin}(V)$ of $\mathrm{SO}(V)$ (and otherwise the root number is -1).

Since there are good formulas to compute the Artin conductor [Ser79, Chapter VI], most of the mystery of the ε-factor lies in the root numbers, and these numbers typically carry deep information. For instance, we saw in Wee Teck's talk that symplectic root numbers are expected to carry information about branching problems for classical groups. In this talk, we will see that orthogonal root numbers carry information about central characters of irreducible admissible representations of G.

The starting point for Gross and Reeder's refinement is to normalize the Haar measure so that the Steinberg representation St_G has formal degree one. Such a measure, called the Poincaré measure, had already been studied by Serre [Ser71]. It satisfies the following properties.

First, for every discrete, cocompact, torsion-free subgroup $\Lambda \subseteq G$, the Euler characteristic of (the rational group cohomology of) Λ is $\chi\big(\mathrm{H}^\bullet(\Lambda, \mathbb{Q})\big) = \mathrm{vol}(\Lambda\backslash G, \mu_G)$. This property was Serre's motivation.

Second, $\mu_G \neq 0$ if and only if $A_G = 1$. So we must assume that $A_G = 1$ for μ_G to be of interest.

Third, the Poincaré measure is a Haar measure up to sign: specifically, $(-1)^{r(G)}\mu_G$ is a Haar measure, where $r(G)$ is the split rank of G.

Fourth, $d(\mathrm{St}_G, \mu_G) = (-1)^{r(G)}$, where St_G is the Steinberg representation [Hum87].

Conjecture 2 ([GR10, Conjecture 7.1(5)]). *In the setting of Result 1, if $A_G = 1$ then*

$$(-1)^{r(G)} \deg(\pi, \mu_G) = \pm \frac{\dim \rho}{|S'_\varphi|} \cdot \frac{\gamma(0, \mathrm{Ad}_G \circ \varphi)}{\gamma(0, \mathrm{Ad}_G \circ \varphi_{\mathrm{prin}})}$$

with sign $\omega(\mathrm{Ad}_G \circ \varphi)/\omega(\mathrm{Ad}_G \circ \varphi_{\mathrm{prin}}) = \pm 1$.

Here φ_{prin} is the principal parameter, the parameter whose L-packet contains the Steinberg representation. This parameter is trivial on W_k and its restriction to the Deligne SL_2 corresponds under the Jacobson-Morozov theorem to the sum of the Lie algebra elements (for a basis of roots) in any pinning of $\widehat{G}$. Let $\underline{Z}$ be the center of $\underline{G}$.

Conjecture 3 ([GR10, Conjecture 8.3]). *In the setting of Result 1,*

$$\frac{\omega(\mathrm{Ad}_G \circ \varphi)}{\omega(\mathrm{Ad}_G \circ \varphi_{\mathrm{prin}})} = \chi_\varphi(z_{\mathrm{Ad}_G}),$$

where $z_{\mathrm{Ad}_G} \in Z$ is a certain involution to be defined momentarily.

Here χ_φ is the character of Z, where $\underline{Z}$ is the center of $\underline{G}$, corresponding to the parameter φ, as originally constructed by Langlands. It is one of Borel's desiderata for the local Langlands correspondence [Bor79, III.10] that χ_φ, which has a simple definition via the correspondence for tori, be the central character of the representations in the L-packet of φ.

2. Orthogonal root numbers

Gross and Reeder proved Result 3 when G is split (and $A_G = 1$ and k is nonarchimedean) using an argument in Galois cohomology. I was able to generalize their theorem by generalizing their proof. This proves Result 3 modulo Borel's desiderata.

Theorem 4 ([Sch21b, Theorem A]). *Let k be a local field, let $r\colon {}^LG \to \mathrm{O}(V)$ be an orthogonal representation, and let $\varphi\colon L_k \to {}^LG$ be a tempered L-parameter. Then*

$$\frac{\omega(r\circ\varphi)}{\omega(r\circ\varphi_{\mathrm{prin}})} = \chi_\varphi(z_r).$$

Since this result lives on the Galois side of the local Langlands correspondence, nothing is lost in assuming that G is quasi-split, and we add this as a standing hypothesis. Here T is a minimal Levi of G and the element $z_r \in Z$ is the involution defined by

$$z_r \overset{\mathrm{def}}{=} \prod_{0<\varpi\in X_*(Z)} \varpi(-1)^{\dim V_\varpi},$$

where V_φ is the ϖ weight space for the action of $\widehat{T}$ on V by r.

Remark 5. I have not defined the root number of a Weil-Deligne representation. But such a definition exists, and for orthogonal representations $\varphi\colon \mathrm{WD}_k \to \mathrm{O}(V)$,

$$\omega(\varphi) = \omega(\varphi|_{W_k}).$$

In particular, $\omega(r\circ\varphi_{\mathrm{prin}}) = \omega(r|_{W_k})$. (Here we use the Weil form of ${}^LG$.)

The key lemma in the proof of the theorem is the following statement in group cohomology.

Lemma 6. *Consider the following commutative diagram, where $c_G \in \mathrm{H}^2_{\mathrm{Borel}}({}^LG, \pi_1(\widehat{G}))$ and $c_{\mathrm{pin}} \in \mathrm{H}^2_{\mathrm{Borel}}(\mathrm{O}(V), \{\pm 1\})$ classify the top and bottom group extensions.*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\widehat{G}) \arrow[r] \arrow[d, "e_r"] & \widehat{G}_{\mathrm{univ}} \rtimes W_k \arrow[r] & {}^LG \arrow[r] \arrow[d, "r"] & 1 & (c_G) \\
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{Pin}(V) \arrow[r] & \mathrm{O}(V) \arrow[r] & 1 & (c_{\mathrm{pin}}).
\end{tikzcd}$$

Then $r^*(c_{\mathrm{pin}}) = e_{r,*}(c_G)\cdot(p^*r|^*_{W_k})(c_{\mathrm{pin}})$ in $\mathrm{H}^2_{\mathrm{Borel}}({}^LG, \{\pm1\})$ where $p\colon {}^LG \to W_k$ is projection.

To prove Result 4, we pull back the conclusion of Result 6 along φ, using in particular that $\mathrm{H}^2(W_k, \{\pm 1\}) \simeq \mathbb{Z}/2\mathbb{Z}$. By Deligne's theorem, the two factors with c_{pin} become the root numbers (normalized by the determinant) of $r\circ\varphi$ and $r\circ\varphi_{\mathrm{prin}}$. To identify $\varphi^* e_{r,*}(c_G)$ with $\chi_\varphi(z_r)$ requires the following computation.

Lemma 7. *Let $\underline{T}$ be a k-torus. Then*

$$\mathrm{H}^2(W_k, X^*(\underline{T})) = \mathrm{Hom}_{\mathrm{cts}}(T^1, \mathbb{C}^\times)$$

where $T^1 \subseteq T$ is the maximal bounded subgroup of T.

Remark 8. In the statement of Result 6, $\mathrm{H}^\bullet_{\mathrm{Borel}}$ denotes the Borel cohomology groups (sometimes called Moore cohomology), a variant of continuous group cohomology defined using Borel-measurable cochains. This generalization is needed because the spin-cover of the orthogonal group does not admit a continuous set-theoretic section. The theory of Borel cohomology was largely worked out in the 60's and 70's by Calvin C. Moore [Moo64a, Moo64b, Moo76a, Moo76b], and it would be very interesting to revisit the subject with modern techniques.

3. Yu supercuspidals

There are several approaches to proving the formal degree conjecture. In Raphaël's talk, for instance, we used twisted endoscopy to deduce the conjecture for classical groups from the conjecture for GL_n. Another approach is to compute everything directly. This approach, while less sophisticated, has the advantage that it produces explicit formulas for the formal degree, which may be of interest in applications. In this final section, I want to explain what happens for Yu supercuspidals [Yu01]. Jessica's talk reviewed the history of these supercuspidals; in particular, every supercuspidal is of the type constructed by Yu if p does not divide the order of the Weyl group of G [Kim07, Fin20].

Theorem 9 ([Sch21a, Theorem A]). *Let $\underline{G}$ be semisimple and let Ψ be a Yu datum with associated supercuspidal representation π. Then*

$$\deg(\pi, \mu) = \frac{\dim \rho}{[G^0_{[y]} : G^0_{y,0+}]} \exp_q \tfrac{1}{2}\left(\dim \underline{G} + \dim \underline{G}^0_{y,0:0+} + \sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\right).$$

The formula of the result is rather complicated. I have not defined several of its constituents because that would require me to explain what a "Yu datum" is. The main takeaway from the formula is that the formal degree is the product of two factors, one coming from depth-zero objects and one coming from positive-depth objects, the latter of which is a power of q. The more "ramified" the representation, the higher its power of q.

To compute the formal degree, we use a general formula for the formal degree of a compactly-induced representation:

$$d(\text{c-Ind}_K^G \sigma, \mu) = \frac{\dim \sigma}{\mathrm{vol}(K, \mu)}.$$

(This is the version of the formula when K is compact-open; in general, one needs a modification in which K is open and compact-mod-center.) Both σ and K derive from the Yu datum. The main difficulty is to compute $\mathrm{vol}(\mu, K)$.

After Yu's construction of supercuspidals, a natural next step was to match the supercuspidals with L-parameters. In a series of recent papers [Kal15, Kal19, Kal21], Kaletha has accomplished this matching in increasing generality for most of Yu's supercuspidals. His most general construction, for the non-singular supercuspidals, matches those whose L-packet does not contain a discrete series representation, at least if p does not divide the order of the Weyl group of G.

Corollary 10 ([Sch21a, Theorem B], [Oha21]). *The formal degree conjecture holds for Kaletha's non-singular L-packets.*

I was able to prove the formal degree conjecture for regular supercuspidals, a slightly less general class of supercuspidals than the non-singular supercuspidals, and Ohara generalized this work to the non-singular supercuspidals.

4. Coda: future work

The formal degree conjecture describes the discrete part of the tempered dual, but the tempered dual is not entirely discrete. Its other components, coming from parabolic inductions of discrete series of Levi subgroups, are finite-index orbifold quotients of real compact tori [Wal03]. Inspired by Langlands's conjecture on Plancherel measures, Hiraga, Ichino, and Ikeda proposed a description of the Plancherel measure on these nondiscrete components of the tempered dual [HII08b, Conjecture 1.4]. I hope to verify their conjecture for the components that come from the parabolic induction of a non-singular Yu supercuspidal.

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