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Flexibilität und symplektische Füllbarkeit

Flexibility and symplectic fillability

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Contents

1. Introduction and motivation

A natural problem to consider in manifold theory is the *filling problem*: is a given n-dimensional smooth manifold the boundary of some compact $(n+1)$-dimensional manifold? While the problem is not entirely trivial (for example, the complex projective space $\mathbb{CP}^2$ is not the boundary of any compact 5-manifold [Fre, Theorem 2.28]), it is the subject of another field called *bordism theory* which is well-established and well-understood.

The problem gets much more interesting when imposing additional structure on the manifolds. In this thesis, we will be concerned about *symplectic manifolds*, i.e. when the $(n+1)$-dimensional manifold carries a *symplectic structure* (see Definition 2.2). Symplectic manifolds are the object of study in *symplectic geometry*. In this case, since symplectic manifolds are even-dimensional, the n-dimensional manifold cannot be symplectic, but is required to carry the odd-dimensional analogue of a symplectic structure, a *contact structure* (see Definition 2.23). This type of filling is called a *symplectic filling*.[1]

There are several different kinds of symplectic fillings, such as strong, weak, Liouville and Stein or Weinstein fillings. In the setting of this thesis, the natural fillings to consider are *Liouville* and *Weinstein fillings* (see Chapter 2).

Before seriously investigating symplectic fillings, one should perhaps study whether there are topological obstructions. For instance, every symplectic manifold admits an almost complex structure (see Definition 2.7 and Proposition 2.11), hence a contact manifold can only be symplectically fillable if it admits a filling with an almost complex structure. However, it turns out that there is no such obstruction: every contact manifold admits a filling with an almost complex structure! We refer the reader to Naomi Kraushar's thesis [Kra16] for a complete proof.

A natural question to consider is the existence and uniqueness of symplectic fillings of a given type. Interestingly, the answers to both questions come from very different methods. This is related to a contrast first observed by Mikhail Gromov [Gro87]: there

[1] We are glossing over a subtle detail: both symplectic and contact manifolds have a natural orientation; for a symplectic filling, the induced orientation of the boundary of the symplectic manifold must agree with the orientation of the contact manifold. One can also define *symplectic caps*, by requiring that the orientations be opposite. The general picture is that symplectic caps add no or little restrictions compared to topological fillings, whereas symplectic fillings *do*. For example, the sphere $\mathbb{S}^n$ can be topologically filled by any closed $(n+1)$-dimensional manifold, by removing a small ball from it and gluing the sphere to the newly created boundary. By Darboux' theorem [MS17, Theorem 3.2.2], an analogous construction works for symplectic manifolds and shows that $\mathbb{S}^{2n-1}$ can be capped by *any* closed symplectic $2n$-dimensional manifold. In contrast, we will see in Chapter 4 that any symplectically aspherical strong symplectic *filling* of $\mathbb{S}^{2n-1}$ is diffeomorphic to the unit ball $\mathbb{D}^{2n}$.

is a distinction between rigidity or “hardness” and flexibility or “softness” which runs at the heart of symplectic geometry.

In many cases, objects in symplectic geometry behave rigidly: their behaviour is severely constrained compared to, for example, the underlying almost complex geometry—the reason is the existence of some abstract invariants which restrict the possible behaviour. In this thesis, we will deal with two of these invariants, *Hamiltonian Floer homology* and its generalisation *symplectic homology*.

In some settings these invariants necessarily vanish and the objects considered behave rather flexibly—for example, symplectic questions are then constrained only by some algebro-topological[2] conditions which must be satisfied. In such cases, one speaks of an h-principle (where h stands for homotopy). In this thesis, we will see that *flexible Weinstein domains* (see Definition 2.131) do satisfy an h-principle; their h-principle in turn rests on an h-principle for so-called *loose Legendrians* (see Section 2.4).

The flexibility and rigidity phenomena work together to create the rich structure found in symplectic geometry today—as a rule of thumb, all interesting questions concern the boundary between these two regimes. Determining and mapping out the boundary between flexibility and rigidity is still a matter of current research, with sometimes surprising insights. The h-principles for loose Legendrians and also for flexible Weinstein domains are such examples.

Aim and main results The aim of this thesis is to explain the result and proof of a theorem by Zhengyi Zhou [Zho18]. The result concerns the question whether a given contact manifold has a unique Liouville filling (see Definition 2.64). It has been known for fifteen years that the answer is No in general—there are contact manifolds, in all dimensions, which admit infinitely many Liouville fillings [OS04; Oba18].

However, there are classes of contact manifolds which seem to remember information about their fillings—so that the filling is determined at least topologically, and sometimes up to diffeomorphism or even up to symplectomorphism. In Chapter 4, we review the relevant literature: for now, let us say that almost all results which apply in higher dimension (i.e. for contact manifolds of dimension at least five) give only information about the diffeomorphism type of Liouville fillings of the contact manifold.

Zhou’s result, refining and essentially building on a preprint by Oleg Lazarev [Laz17], goes beyond that: Zhou shows that if a contact manifold of dimension at least five admits a flexible Weinstein filling (see Definition 2.135), all its topologically simple (see Definition 5.2) Liouville fillings have trivial symplectic homology (see Section 3.3). This doesn’t go as far as establishing uniqueness of the fillings, but it shows that their symplectic homology vanishes, which provides evidence towards their uniqueness. (A flexible Weinstein filling is a special case of a topologically simple Liouville filling; one can show that it always has trivial symplectic homology.)

[2] This is the adjective corresponding to algebraic topology.

Structure of this thesis All material we present is known; we make no claims of originality. We begin by presenting the two concepts needed to understand the statement of the main theorem.

In Chapter 2, we introduce the concepts of Liouville and Weinstein manifolds, which are the setting we are working in. We explain what flexible Weinstein domains are and state their h-principle, which is essential for the proof of Zhou's result. We also define the concepts of surgery and handle attachment which we need later.

In Chapter 3, we define symplectic homology, the invariant used in Zhou's main theorem. We begin by presenting the definition of Hamiltonian Floer homology, and explain how this can be generalised to define symplectic homology. The proof of Zhou's result uses several key properties of symplectic homology (such as an exact sequence and a ring structure), which we also explain.

After these preparations, in Chapter 4 we state the main theorem and outline how it extends previous research. Zhou's proof follows the same overall outline as a previous proof, but requires significant new insights to carry out the overall plan. The main technical advance concerns a class of *asymptotically dynamically convex* (Definition 5.8) contact manifolds, which have two crucial properties that make the proof work. In Chapter 5, we explain and motivate their definition and outline their crucial properties. In Chapter 6, we give a proof of Zhou's main result, combining the algebraic properties of symplectic homology with the special properties of ADC manifolds. In Chapter 7, we reflect on this proof and look at next steps for research.

This thesis is based on a lot of material, and there is not enough space to explain everything. I have attempted to explain the essential concepts in detail, to give all the high-level ingredients and to outline how these fit together. Some technical points will be skipped; others are only sketched. Nevertheless, when skipping something I have tried my best to give precise statements of the result used, and to phrase the result in a way that is useful in a wider context than just this thesis.

2. Liouville and Weinstein manifolds

In this chapter, we present the concepts of a Liouville and (flexible) Weinstein domain, and their associated fillings. Almost all material in Section 2.1 can be found in either or both of McDuff and Salamon's [MS17] and Geiges' [Gei08] textbooks. Most of the material in Sections 2.2 through 2.6 is taken from Cieliebak and Eliashberg's book [CE12].

2.1. Symplectic and contact manifolds

We begin by reviewing the very basic definitions underlying everything written thereafter: those of symplectic and contact manifolds. For space reasons, not everything will be shown in detail; henceforth a proof symbol will denote either the end or the lack of a proof. In the latter case, we will give a reference for the result.

Definition 2.1. *A* symplectic vector space *is a pair* (V, ω) *of a finite-dimensional* $\mathbb{R}$-*vector space* V *and a non-degenerate skew-symmetric bilinear form* $\omega\colon V \times V \to \mathbb{R}$.

Definition 2.2. *A* symplectic structure *on a smooth manifold* M *is a closed* 2*-form on* M *which is also non-degenerate, i.e. at each point* $p \in M$*, the bilinear form* $\omega_p\colon T_pM \times T_pM \to \mathbb{R}, (X, Y) \mapsto \omega_p(X, Y)$ *is non-degenerate. The form* ω *is called a* symplectic form *on* M*. A* symplectic manifold *is a pair* (M, ω) *of a smooth manifold* M *and a symplectic form* ω *on* M*.*

Remark. Equivalently, a closed 2-form ω on a smooth manifold M is symplectic if and only if for each $p \in M$, the tangent space (T_pM, ω_p) is a symplectic vector space.

In general, it is a delicate question whether a given smooth manifold admits a symplectic structure. Two elementary necessary conditions are the following.

Proposition 2.3 ([MS17, p. 94])**.** *A symplectic manifold is even-dimensional and has a natural orientation.* $\square$

Orientability part follows from the following result, which we state explicitly since we will use twice more in this section.

Lemma 2.4 ([MS17, Corollary 2.1.4])**.** *For a* 2*-form* ω *on a* $2n$*-dimensional smooth manifold* M*, the* n*-fold exterior power* $\omega^n = \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}}$ *is a volume form on* M *if and only if* ω *is non-degenerate. In particular, if* (M, ω) *is symplectic, then* ω^n *is a volume form.* $\square$

For closed manifolds, there are further topological conditions.

Proposition 2.5. *If (M, ω) is a* closed *$2n$-dimensional symplectic manifold, the form ω is never exact.*

Proof. This follows from Stokes' theorem (which can be applied since M is closed). Assume $\omega = d\lambda$ were exact, then ω^n were also exact since $\omega^n = (d\lambda)^n = d(\lambda \wedge (d\lambda)^{n-1})$.

By Lemma 2.4, the form ω^n is a volume form, hence we obtain a contradiction from

$$0 < \operatorname{vol}(M) = \int_M \omega^n = \int_M d(\lambda \wedge (d\lambda)^{n-1}) = \int_{\partial M} \lambda \wedge (d\lambda)^{n-1} = 0. \qquad \square$$

In later sections, we will encounter exact symplectic manifolds, which consequently will not be closed any more.

In exactly the same way, one can show the following result, which shows that, e.g., spheres $\mathbb{S}^n$ for $n > 2$ never admit a symplectic structure.

Proposition 2.6. *If (M, ω) is a* closed *$2n$-dimensional symplectic manifold, each de Rham cohomology group $H^{2k}(M)$ for $1 \leq k \leq n-1$ does not vanish.* $\square$

Both necessary conditions in Proposition 2.3 follow from a much deeper result: every symplectic manifold admits an almost complex structure.

We explain this concept for general vector bundles, since we will use a similar result for contact manifolds (see Definition 2.23 below) and this definition allows treating both settings on the same footing.

Definition 2.7. *Let $E \to M$ be a smooth vector bundle. A* complex (bundle) structure *on E is a smooth family J of fibre-preserving linear maps $J_p\colon E_p \to E_p$ such that $J^2 = -\operatorname{id}$, i.e. $J_p \circ J_p = -\operatorname{id}_{E_p}$ for all $p \in M$. Formally speaking, smoothness means that the map $p \mapsto J_p$ is a smooth section of the endomorphism bundle* $\operatorname{End}(E)$. *An* almost complex structure *on a smooth manifold is a complex structure on its tangent bundle.*

Since an almost complex structure gives any vector bundle the structure of a complex vector bundle, the conditions in Proposition 2.3 follow from the existence of almost complex structures on any symplectic manifold.

In many applications (including Hamiltonian Floer homology in Section 3.1), the space of *all* almost complex structures is too vast to be useful. Instead, one considers the class of *compatible* almost complex structures.[1] Again, this definition can be made in the general context of vector bundles: we just have to generalise symplectic manifolds appropriately.

[1] There is also a weaker condition, of *tame* almost complex structures. This sometimes has advantages since the set of tame almost complex structures is an open subset, but some applications become much simpler when using compatible almost complex structures.

Definition 2.8. *A* symplectic structure ω *on a smooth vector bundle* $E \to M$ *is a smooth family of symplectic structures* ω_p *on each fibre* E_p*. Formally, smoothness means that the map* $p \mapsto \omega_p$ *is a smooth section of the bundle* $\Lambda^2 E^* \to M$*, the second exterior power of the dual bundle of* E*. A* symplectic vector bundle *is a pair* (E, ω) *of a smooth vector bundle* $E \to M$ *and a symplectic structure* ω *on* E*.*

Remark 2.9. A symplectic structure on a smooth manifold M is a particular symplectic structure on the tangent bundle TM, hence the tangent bundle of every symplectic manifold is a symplectic vector bundle. For contact manifolds, we obtain a symplectic vector bundle in a different way (see Lemma 2.29).

Definition 2.10. *A complex structure* J *on a symplectic vector bundle* (E, ω) *over* M *is called* compatible *with* ω *if and only if for each* $p \in M$*, the bilinear form* $g_p := \omega_p(\cdot, J_p(\cdot))$ *defines an inner product on the fibre* E_p*. In other words,* $g := \omega(\cdot, J\cdot)$ *defines a Euclidean structure on* E*. In particular, an* ω*-compatible almost complex structure* J *on a symplectic manifold* (M, ω) *defines a Riemannian metric* $g := \omega(\cdot, J\cdot)$ *on* M*.*

The following result is crucial for the further theory and goes back to Gromov [Gro85].

Proposition 2.11 ([Gei08, Proposition 2.4.5])**.** *Let* (E, ω) *be a symplectic vector bundle over a manifold* M*. Then the space* $\mathcal{J}(E)$ *of* ω*-compatible complex structures on* E *is non-empty and contractible.* $\square$

Remark 2.12. Here, we endow the space $\mathcal{J}(E)$ with the C^∞_{loc}-topology, i.e. a sequence $J_k \in \mathcal{J}(E)$ converges if and only if it is C^∞-convergent on all compact subsets. This topology is also known as *weak* C^∞*-topology*; see e.g. [Hir76, Chapter 2] for details.

Hence, every symplectic manifold admits a compatible almost complex structure. This is another (much deeper) necessary condition for the existence of a symplectic structure, and in fact implies the conditions in Proposition 2.3.

There are various natural concepts of submanifolds of a symplectic manifold. Since they are defined point-wise, they also make sense for symplectic vector spaces.

Definition 2.13. *Let* (V, ω) *be a symplectic vector space and* $Y \subset V$ *be a linear subspace. The subspace* Y *is called*

- symplectic *if and only if the restriction* $\omega|_Y$ *is non-degenerate, i.e. if and only if* $(Y, \omega|_Y)$ *is a symplectic vector space again.*
- isotropic *if and only if* ω *vanishes on* Y*, i.e. if and only if* $\omega|_Y \equiv 0$*.*
- Lagrangian *if and only if* Y *is isotropic and* $\dim Y = n$*.*

Remark. As this definition may suggest, any isotropic subspace of a given symplectic vector space is contained in a Lagrangian subspace, i.e. Lagrangian subspaces are the maximal isotropic subspaces (with respect to the natural partial order by inclusion). This can be shown by applying Zorn's lemma, but there is also a more conceptual proof that does not use Zorn's lemma.

Definition 2.14. *For a subspace $Y \subset V$ of a symplectic vector space (V, ω), we define its* symplectic complement Y^ω *as the set*

$$Y^\omega = \{v \in V : \omega(v, u) = 0 \text{ for all } u \in Y\}.$$

In Section 2.5, we will also use the following result.

Lemma 2.15 ([Gei08, Lemma 1.3.3])**.** *Let (V, ω) be a symplectic vector space and $U \subset V$ an isotropic subspace. Then ω induces a symplectic structure on the quotient U^ω/U.* □

These definitions transfer to symplectic manifolds by considering them at each point.

Definition 2.16. *Let (M, ω) be a $2n$-dimensional symplectic manifold and $N \subset M$ be a smooth submanifold. The submanifold N is called*

- symplectic *if and only if each $T_pN \subset T_pM$ is a symplectic vector space, i.e. if and only if the restriction $\omega|_N$ defines a symplectic form on N.*
- isotropic *if and only if each $T_pN \subset T_pM$ is an isotropic vector space, i.e. if and only if for each $p \in N$, the restriction $\omega_p|_{T_pN}$ vanishes.*
- Lagrangian *if and only if each subspace $T_p \subset T_pM$ is Lagrangian, i.e if and only if N is isotropic and* $\dim N = n$.

In these cases, we call N a symplectic (isotropic, Lagrangian) submanifold of (M, ω).

The dimension of an isotropic submanifold is severely restricted, as the following proposition states.

Proposition 2.17. *If $N \subset (M, \omega)$ is an isotropic submanifold, then $\dim N \leq \frac{1}{2} \dim M$.*

Proof sketch. There is a dimension formula for the symplectic complement of any subspace $Y \subset V$:

$$\dim Y + \dim Y^\omega = \dim V.$$

See [MS17, Lemma 2.1.1] for details. Now the result follows from observing that N is an isotropic submanifold if and only if each tangent space $T_pN \subset T_pM$ for $p \in N$ satisfies the condition $T_pN \subset (T_pN)^\omega$. □

The most natural notion of equivalence between symplectic manifolds is the following.

Definition 2.18. *A diffeomorphism $\phi \colon M \to N$ between symplectic manifolds (M, ω_1) and (N, ω_2) is called a* symplectomorphism *if and only if one has $\phi^*\omega_2 = \omega_1$. In this case, the manifolds (M, ω_1) and (N, ω_2) are called* symplectomorphic.

In this thesis, we will encounter another approach for defining equivalence: we have several notions of equivalence by deformation. The simplest one is for symplectic manifolds, where it is called *symplectic deformation*.

Definition 2.19. *Two symplectic structures ω and ω' on a smooth manifold M are called* deformation equivalent *if and only if there is a smooth family of symplectic forms $\{\omega_t\}_{t\in[0,1]}$ on M such that $\omega_0 = \omega$ and $\omega_1 = \omega'$. Two symplectic manifolds (M, ω) and (M', ω') are called* deformation equivalent *if and only if there exists a diffeomorphism $\phi\colon M \to M'$ such that ω and $\phi^*\omega'$ are deformation equivalent.*

In odd dimension, there are no symplectic manifolds, but there is the related notion of a contact manifold. Their definition requires two technical concepts.

Definition 2.20. *A* smooth hyperplane field *on a smooth m-dimensional manifold M is a collection $\{\xi_p\}_{p\in M}$ of $m-1$-dimensional linear subspaces $\xi_p \subset T_pM$ which "vary smoothly with p". Smoothness means that the union $\bigcup_{p\in M} \xi_p \subset TM$ forms a smooth subbundle of rank $(m-1)$ of the tangent bundle TM; equivalently, any point $p \in M$ has a neighbourhood $U \ni p$ on which there are smooth vector fields $X_1, \ldots, X_{m-1}\colon U \to TM$ such that at each point $q \in U$, the vectors $X_1(q), \ldots, X_{m-1}(q) \in T_qM$ form a basis of ξ_q.*

Definition 2.21. *A smooth hyperplane field ξ on a smooth manifold M is called* co-oriented *if and only if there exists a 1-form α on M such that $\xi_p = \ker(\alpha_p)$ for all $p \in M$.*

Example 2.22. For $\mathbb{R}^{2n-1}$ with coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z)$, the smooth 1-form $\alpha := dz + \sum_{i=1}^{n-1} x_i dy_i$ defines a smooth hyperplane field $\xi = \ker \alpha$. We compute $d\alpha = \sum_i dx_i \wedge dy_i$ and thus

$$\alpha \wedge (d\alpha)^{n-1} = (n-1)!\, dz \wedge dx_1 \wedge dy_1 \wedge \ldots \wedge dx_{n-1} \wedge dy_{n-1} \neq 0,$$

since $dx_1 \wedge dy_1 \wedge \ldots \wedge dx_{n-1} \wedge dy_{n-1} \wedge dz$ is the standard volume form on $\mathbb{R}^{2n-1}$.

Definition 2.23. *Let M be a smooth manifold of dimension $2n-1$. A* contact structure *on M is a co-oriented smooth hyperplane field $\xi = \ker \alpha \subset TM$ on M which satisfies the condition $\alpha \wedge (d\alpha)^{n-1} \neq 0$, i.e. the form $\alpha \wedge (d\alpha)^{n-1}$ is non-zero at each point. The 1-form α is called a* contact form *for ξ. The pair (M, ξ) is called a* contact manifold.

The condition $\alpha \wedge (d\alpha)^{n-1}$ is often referred to as ξ being "maximally non-integrable".

Remark 2.24. One can show (see e.g. [Gei08, Lemma 1.1.1]) that any smooth hyperplane field can be locally co-oriented, i.e. every point $p \in M$ has an open neighbourhood $U \ni p$ and a 1-form α on U such that $\xi_p = \ker \alpha_p$ for all $p \in U$. There need not exist a global co-orientation. In this thesis, however, we only consider co-oriented contact structures.

Remark 2.25. A contact form α for a $(2n-1)$-dimensional contact manifold (M, ξ) induces an orientation, via the form $\alpha \wedge (d\alpha)^{n-1}$. If M has a prescribed orientation, a contact form α is called *positively oriented* if and only if this orientation matches the orientation induced by the volume $\alpha \wedge (d\alpha)^{n-1} > 0$.

Example 2.26. The 1-form α on $\mathbb{R}^{2n-1}$ from Example 2.22 is a contact form, hence $\xi = \ker \alpha$ is a contact structure on $\mathbb{R}^{2n-1}$, called the *standard contact structure* on $\mathbb{R}^{2n-1}$. It is positively oriented w.r.t. the standard orientation of $\mathbb{R}^{2n-1}$.

Example 2.27. Consider an odd-dimensional sphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{R}^{2n}$ and the smooth 1-form $\alpha := \sum_{i=1}^{n}(x_j dy_j - y_j dx_j)$. Let $r := \sqrt{\sum_i x_i^2 + \sum_i y_i^2}$ be the radial coordinate on $\mathbb{R}^{2n}$. One easily computes that

$$r\, dr \wedge \alpha \wedge (d\alpha)^{n-1} = r^2\, (n-1)!\, dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n,$$

hence α is a contact form on the non-zero level sets of r, including on $\mathbb{S}^{2n-1}$. The form α is positively oriented w.r.t. the standard orientation on $\mathbb{S}^{2n-1}$. Thus, $(M, \xi = \ker \alpha)$ is a contact manifold; ξ is called the *standard contact structure* on $\mathbb{S}^{2n-1}$.

In general, there is a large space of contact forms for a given contact structure. We can view a contact structure as an equivalence class of 1-forms. Sometimes, the choice of contact form matters. For example, in Section 3.3, it will be instrumental to choose a contact form α which has a certain property.

Lemma 2.28 ([MS17, Proposition 3.5.1])**.** *Let α and α' be 1-forms on a smooth manifold M with $\xi = \ker \alpha = \ker \alpha'$. Then, there exists a non-zero smooth function $f\colon M \to \mathbb{R}$ with $\alpha' = f\alpha$. Hence, α is a contact form if and only if α' is. Moreover, the symplectic structures $d\alpha$ and $d\alpha'$ on ξ are related by $d\alpha'|_\xi = f\, d\alpha|_\xi$.* □

In particular, we observe that the compatibility of a complex structure $J \in \mathcal{J}(\xi, d\alpha)$ is independent of the choice of contact form α as long as the resulting orientation of M is prescribed.

The reader may wonder how symplectic and contact manifolds are related. There are several such relations, the first of which is the following.

Lemma 2.29. *Let $(M, \xi = \ker \alpha)$ be a $(2n-1)$-dimensional contact manifold. For each $p \in M$, the space $(\xi_p, d\alpha_p)$ is a symplectic vector space. In fact, $(\xi, d\alpha)$ is a symplectic vector bundle over M.*

Proof. By definition, ξ is a smooth vector bundle. Since $\alpha \wedge (d\alpha)^{n-1} \neq 0$ and $\xi = \ker \alpha$, we have $(d\alpha)^{n-1} \neq 0$ on ξ, hence $d\alpha|_\xi$ is non-degenerate by Lemma 2.4. The forms $d\alpha_p$ vary smoothly since α is smooth. □

The reader may wonder whether a stronger statement can be made, such as whether ξ is the tangent bundle of some submanifold. The answer is no, and this is where the condition of ξ being "maximally non-integrable" comes in.

Definition 2.30. *Let $\xi \subset TM$ be a smooth hyperplane field on M. A submanifold $N \subset M$ is called* integral *for ξ if and only if $T_pN = \xi_p$ for all $p \in N$. The hyperplane field ξ is called* integrable *if and only if every point $p \in M$ is contained in an integral submanifold for ξ.*

In other words, we could wonder whether a contact structure ξ can be integrable. A general criterion for determining this is given by the Frobenius integrability theorem.

Theorem 2.31 (Frobenius integrability theorem, e.g. [Lee02, Theorem 19.10])**.** *A smooth hyperplane field ξ on a smooth manifold M is integrable if and only if local sections of ξ are closed under the Lie bracket.*

Observation. If a contact structure $\xi = \ker\alpha$ were integrable, the Lichnerowicz rule

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \tag{2.1}$$

would imply that $d\alpha$ vanishes for vectors in ξ, i.e. one would have $\alpha \wedge d\alpha = 0$. The condition $\alpha \wedge (d\alpha)^{n-1}$ in Definition 2.23 is as far from that as possible, hence the term "maximally non-integrable" hyperplane field.

Hence, there is no $(2n-1)$-dimensional submanifold $L \subset M$ with $T_pL = \xi_p$ for all $p \in L$. The weaker condition $T_pL \subset \xi_p$ can be satisfied, but only for submanifolds of dimension at most $n-1$.

Definition 2.32. *A submanifold L of a $(2n-1)$-dimensional contact manifold $(M, \xi = \ker\alpha)$ is called* isotropic *if and only if $T_pL \subset \xi_p$ for all $p \in L$. If in addition $\dim L = n-1$, it is called* Legendrian*.*

Proposition 2.33. *Let $(M, \xi = \ker\alpha)$ be a $(2n-1)$-dimensional contact manifold and $L \subset M$ be an isotropic submanifold. Then T_qL is an isotropic subspace of $(\xi_q, d\alpha_q)$ for each $q \in L$. In particular, we have $\dim L \leq n-1$.*

Proof. If $X, Y \in T_pL$ are tangent vectors at $p \in L$, we have $\alpha(X) = 0 = \alpha(Y)$ and also $d\alpha_p([X, Y]_p) = 0$ since L is integral. Thus, equation (2.1) implies $d\alpha_p(X, Y) = 0$. Hence, $d\alpha$ vanishes on L. □

Finally, we also mention the natural definition of morphism of contact structures. For that, we emphasize that the contact structure is really given by the hyperplane field ξ. Usually, we want to also preserve the orientation of ξ.

Definition 2.34. *Let $(M, \xi = \ker\alpha)$ and $(N, \eta = \ker\beta)$ be contact manifolds. A smooth map $\phi\colon M \to N$ is called a* contactomorphism *if and only if $\phi^*\beta = f\alpha$ for some positive smooth function f on M.*[2] *The map ϕ is called a* strict contactomorphism *if and only if the same holds for $f \equiv 1$. In this case, (M, ξ) and (N, η) are called* (strictly) contactomorphic*. In particular, a contactomorphism is orientation-preserving.*

A contact form determines a special vector field, the Reeb vector field.

Lemma/Definition 2.35 ([Gei08, Lemma/Definition 1.1.9])**.** *Let (M, ξ) be a contact manifold. For a contact form α, there is a unique vector field R_α on M, called the* Reeb vector field *for α, which satisfies the conditions $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$.* □

[2] While our terminology is the commonly used one, note that Geiges' book uses these terms differently.

Note that the Reeb vector fields for different contact forms α and $f\alpha$ may have very different properties. We will exploit that in Section 3.3. In that section, we will also meet its close cousin, the Hamiltonian vector field.

There is another relation between contact and symplectic manifolds: every contact manifold can be converted into a symplectic manifold by a construction called *symplectisation*.

Definition 2.36 ([MS17, Proposition 3.5.23])**.** *Let $(M, \xi = \ker\alpha)$ be a contact manifold. Consider the product manifold $\mathbb{R} \times M$ and the 1-form $e^r\alpha := (e^r\alpha)_{(r,p)} = e^r\alpha_p$. The* symplectisation *of $(M, \xi = \ker\alpha)$ is the manifold $\mathbb{R}\times M$ together with the 2-form $d(e^r\alpha)$.*

Proposition 2.37. *The form $d(e^r\alpha)$ is a symplectic form on $\mathbb{R} \times M$, hence the symplectisation of a contact manifold is a symplectic manifold. Each level set $(\{r\} \times M, \alpha)$ of the symplectisation $(\mathbb{R} \times M, d(e^r\alpha))$ is a contact manifold.* □

Observation 2.38 ([MS17, Proposition 3.5.23])**.** The symplectisation $(\mathbb{R} \times M, d(e^r\alpha))$ of (M, ξ) is independent of the choice of contact form: for any two positively oriented contact forms α, α' for (M, ξ), the resulting manifolds $(\mathbb{R} \times M, d(e^r\alpha))$ and $(\mathbb{R}\times M, d(e^r\alpha'))$ are symplectomorphic. □

Remark 2.39. In the literature, there are several variants of this definition, differing in signs and other conventions. For example, some authors (including Lazarev [Laz17]) put the factor M first, which is *wrong*: unless one takes further contortions (which Lazarev doesn't), this gives the wrong orientation to the symplectisation. See [Wen15] for details. We take the liberty to just correct this.
One can also use the convention $((0,\infty) \times M,\ d(r\alpha))$, which poses no problems. See Appendix A.1 for a collection of the sign conventions we use in this document. Note that the definitions in [MS17] use different sign conventions.

It is an easy exercise to show that two contactomorphic contact manifolds have symplectomorphic symplectisations. The converse statement was an open problem for a long time, but now has a negative answer.

Theorem 2.40 ([Cou14; Cou16])**.** *In any dimension $2n - 1 \geq 3$, there are contact manifolds which are not contactomorphic, but have exact symplectomorphic[3] symplectisations.* □

When defining symplectic homology, we will need another basic concept of the *first Chern class*. Since explaining this in all detail would take far too long, we contend ourselves to the following result.

Lemma/Definition 2.41 ([MS17, Section 2.7])**.** *One can associate to every complex vector bundle E over a smooth manifold N a cohomology class $c_1(E) \in H^2(N)$, called the* first Chern class *of the bundle E, such that*

[3]See Definition 2.52 in the next section.

(i) for all smooth maps $f\colon M \to N$, one has $c_1(f^*E) = f^*c_1(E)$, and

(ii) $c_1(E \oplus F) = c_1(E) + c_1(F)$.

(iii) If $N =: \Sigma$ is a closed oriented surface and $E \to \Sigma$ a line bundle, the number $\int_\Sigma c_1(E) = \langle c_1(E), [\Sigma]\rangle \in \mathbb{Z}$ is called the first Chern number *of E. Here, $[\Sigma] \in H_2(\Sigma)$ is the* fundamental class *of Σ. If $s\colon \Sigma \to E$ is a section of E and transverse to the zero section, the first Chern number $c_1(E)$ equals the number of zeroes of s, counted with signs. See [MS17, Theorem 2.7.5].*

In the first item, $f^*E \to M$ is the pullback of the bundle E via f, and $f^*c_1(E)$ is the pullback of the homology class $c_1(E)$ under the map f. In other words, the first Chern class is compatible with pullbacks and direct sums.
In particular, the first Chern class is an invariant of complex vector bundles.

As we already saw, any symplectic vector bundle becomes a complex vector bundle after choosing a compatible complex structure. Since both symplectic and contact manifolds have a symplectic vector bundle naturally associated to them, we can try to define the first Chern class of a symplectic or contact manifold by choosing any compatible almost complex structure.

Definition 2.42. *For a symplectic manifold (M, ω), the* first Chern class *$c_1(M, \omega)$ of (M, ω) is defined as $c_1(M, \omega) := c_1(TM, J)$, where $J \in \mathcal{J}(TM, \omega)$ is any compatible almost complex structure. For a contact manifold $(M, \xi = \ker\alpha)$, the* first Chern class *$c_1(M, \xi)$ of (M, ξ) is defined as $c_1(M, \xi) := c_1(\xi, J)$, where $J \in \mathcal{J}(\xi, d\alpha)$ is any compatible almost complex structure.*

The natural question is whether this is well-defined. Fortunately, using Proposition 2.11, one can show that the answer is yes. Recall that two choices for the contact form α yield isomorphic symplectic vector bundles; hence our back is covered by the following result.

Proposition 2.43 ([Che07, Theorem 2.7])**.** *Given a symplectic vector bundle (E, ω) over a smooth manifold M and two compatible almost complex structures $J_1, J_2 \in \mathcal{J}(E)$, the vector bundles (E, J_1) and (E, J_2) are isomorphic as complex vector bundles.* □

2.2. Liouville manifolds and domains

In this thesis, we will mostly deal with a special type of symplectic manifolds, called *Liouville manifolds*. In addition to a symplectic form, they have two equivalent pieces of extra structure, either a Liouville form or a Liouville vector field.

Definition 2.44. *A* Liouville form *on a smooth manifold M is a 1-form λ on M such that $\omega := d\lambda$ is a symplectic form.*

Since a symplectic form is non-degenerate, Liouville forms give rise to a dual notion, of a *Liouville vector field*.

Proposition 2.45. *Let λ be a Liouville form on a smooth manifold M, denote $\omega = d\lambda$. There is a unique vector field X which satisfies the relation $\iota_X \omega = \lambda$, i.e. $\omega(X, \cdot) = \lambda$. The vector field X is called the* Liouville vector field *of λ.* □

Definition 2.46. *An* exact symplectic manifold *is a pair (V, λ) of a smooth manifold V and a Liouville form λ on V. Equivalently, an exact symplectic manifold is a triple (V, ω, X) consisting of a symplectic manifold (V, ω) and a vector field X satisfying $\mathcal{L}_X \omega = \omega$.*

Remark 2.47. More precisely, these two definitions correspond via the mappings

$$\begin{aligned} (V, \lambda) &\longmapsto (V, \omega = d\lambda, X \text{ defined by } \iota_X \omega = \lambda) \\ (V, \lambda := \iota_X \omega) &\longleftarrow (V, \omega, X) \end{aligned}$$

To check that these are well-defined, we use Cartan's formula and that ω is closed: in direction "$\mapsto$", we see that

$$\mathcal{L}_X \omega = \iota_X \omega + d\iota_X \omega = 0 + d\lambda = \omega;$$

for the direction "$\leftarrow$", we perform the same computation in reverse:

$$d\lambda = d\iota_X \omega = d\iota_X \omega + \iota_X (d\omega) = \mathcal{L}_X \omega = \omega.$$

It is easy to check that these mappings are mutually inverse.

A slightly confusing point in the literature is that Liouville vector fields can also be defined without recourse to Liouville forms: in that case, the definition is the following.

Definition 2.48. *A* Liouville vector field *on a symplectic manifold (V, ω) is a smooth vector field X which satisfies $\mathcal{L}_X \omega = \omega$.*

There is no ambiguity about the definitions: by Remark 2.47, X is a Liouville vector field for the exact symplectic manifold (V, λ) if and only if it is one for the symplectic manifold $(V, \omega = d\lambda)$. A more subtle point is that the space of Liouville vector fields as in Definition 2.48 is infinite-dimensional in general, and similarly that an exact symplectic form ω has *many* possible primitives λ (if $\omega = d\lambda$, the form $\lambda' = \lambda + df$ for any smooth function f is another primitive). However, in the definition of an exact symplectic manifold, we have already *chosen* a specific λ, so that the Liouville vector field for an exact symplectic manifold is uniquely determined by its definition.

Observation 2.49. Let (V, ω, X) be an exact symplectic manifold. The set of Liouville vector fields on the symplectic manifold (V, ω) is an infinite-dimensional linear space.

Proof. Observe that if X is a Liouville vector field on (V, ω), so is $X + X_H$, where X_H is the Hamiltonian vector field (see Definition 3.1) of any smooth function $H: V \to \mathbb{R}$. Then apply Observation 3.3. $\square$

The condition $\mathcal{L}_X\omega = \omega$ implies that a symplectic form behaves nicely under the flow of a Liouville vector field.

Proposition 2.50. *Let X be a Liouville vector field for the Liouville form λ on V. Then the flow ϕ_t of X "dilates" the symplectic form $\omega = d\lambda$: one has $(\phi_t)^*\omega = e^t\omega$ whenever ϕ_t is defined.*

Proof sketch. Given any $p \in V$ and $X, Y \in T_pV$, consider the smooth function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(t) := ((\phi_t)^*\omega)_p(X, Y)$. For any $t \in \mathbb{R}$, using the group law for the flow ϕ_t and the identity $\mathcal{L}_X\omega = \omega$, we compute

$$\begin{aligned} f'(t) &= \partial_s f(s)|_{s=t} = \partial_s\, \phi_s^*\omega(X, Y)|_{s=t} = \phi_t^*(\partial_{\tilde{s}}\, \phi_{\tilde{s}}^*\omega(X, Y)|_{\tilde{s}=0}) \\ &= \phi_t^*(\mathcal{L}_X\omega(X, Y)) = \phi_t^*\omega(X, Y) = f(t). \end{aligned}$$

This differential equation implies that $f(t) = e^t f(0)$, hence the claim follows. $\square$

Because of this observation, one calls the Liouville vector X *symplectically expanding* and its negative $-X$ *symplectically contracting.*

Proposition 2.51. *On an exact symplectic manifold (V, ω, X) with $\lambda := \iota_X\omega$, we have $\mathcal{L}_X\lambda = \lambda$, hence the flow ϕ_t of X also expands the Liouville form λ, i.e. $(\phi_t)^*\lambda = \lambda$ whenever ϕ_t is defined.*

Proof. By definition of λ resp. X, we have $\iota_X\lambda = d\lambda(X, X) = 0$ resp. $\iota_X(d\lambda) = \lambda$, hence Cartan's formula implies $\mathcal{L}_X\lambda = \lambda$. The remaining proof is the same as for Proposition 2.50. $\square$

The natural maps between exact symplectic manifolds are not symplectomorphisms, but are required to satisfy a slightly stronger condition.

Definition 2.52. *A map $\Phi: (V_1, \omega_1, X_1) \to (V_2, \omega_2, X_2)$ between exact symplectic manifolds is called* exact symplectic *or an* exact symplectomorphism *if and only if the 1-form $\Phi^*\lambda_2 - \lambda_1$ is exact, where $\lambda_i := \iota_{X_i}\omega_i$ for $i = 1, 2$.*

Remark 2.53. An exact symplectomorphism $\phi: (V, \omega, X) = (V', \omega', X')$ does not preserve the associated primitives on V resp. V', but the pullback $\phi^*\omega'$ of the symplectic form on V' is still exact. For all later applications, that is sufficient.

Finally, we get to the definition of Liouville manifolds.

Definition 2.54. *A* Liouville manifold *is an exact symplectic manifold (V, ω, X) with the properties*

- *the expanding vector field X is complete,*
- *the manifold V is* convex*, in the sense that there each connected component V_i of V has an exhaustion $V_i = \bigcup_{k=1}^{\infty} V_i^k$ by nested compact domains $V_i^k \subset V_i$ with smooth boundaries along which X is outward pointing.*

In this case, we call (ω, X) a Liouville structure *on V.*

Remark 2.55. By the second condition, a Liouville manifold has no boundary. Since every connected smooth manifold can be exhausted by compact domains with smooth boundary, the crucial part of the second condition is that X be outward pointing along each boundary ∂V_i^k.

The compact analogues of Liouville manifolds are called *Liouville domains*. By Proposition 2.5, they must have non-empty boundary. The correct definition contains an additional condition on the Liouville vector field at the boundary, which is motivated by the following result.

Lemma 2.56 ([Gei08, Lemma 1.4.5])**.** *Let (M, ω) be a symplectic manifold, $S \subset M$ be a hypersurface*[4] *and X a Liouville vector field defined in a neighbourhood of S. The 1-form $\alpha := \iota_X \omega$ is a contact form on S if and only if X is transverse to S; by the latter, we mean that $X_p \notin T_p S$ for all $p \in S$ ("X is nowhere tangent to S").* □

Hence, if X is transverse to the boundary of a Liouville domain, the boundary has an induced contact structure. This is a natural condition to impose since contact manifolds are in many ways the odd-dimensional analogue of symplectic manifolds (for example, they also enjoy analogues of Darboux' and Moser's theorem, which are two classical results for symplectic manifolds). A priori, the vector field could be outward or inward pointing at the boundary. An easy application of Stokes' theorem shows that there are some restrictions.

Observation 2.57. There is no compact exact symplectic manifold (W, ω, X) such that X is inward-pointing along ∂W. □

Depending on whether there is some region in which X is inward-pointing, we arrive at the definitions of a Liouville cobordism and Liouville domain.

Definition 2.58. *A* cobordism *W is a compact oriented smooth manifold W with oriented boundary ∂W which decomposes as $\partial W = \partial_+ W \cup \partial_- W$, where the orientation agrees with the boundary orientation for $\partial_+ W$ and is opposite to it for $\partial_- W$. One or both of $\partial_\pm$ may be empty.*

Definition 2.59. *A* Liouville cobordism *(W, ω, X) is a compact cobordism W with an exact symplectic structure (ω, X) such that X points outwards along $\partial_+ W$ and inwards along $\partial_- W$. A Liouville cobordism W with $\partial_- W = \emptyset$ is called a* Liouville domain*. We call (ω, X) a* Liouville structure *on W.*

[4] By definition, a *hypersurface* in a manifold is a submanifold of codimension 1.

Corollary 2.60. *If (W, ω, X) is a Liouville domain, then $\alpha := \lambda|_{\partial W}$ defines a contact form on ∂W.* □

Observe that the second property in Definition 2.54 is precisely that a Liouville manifold can be exhausted by a sequence of Liouville domains. What is more, every Liouville domain W can be turned into a Liouville manifold $\widehat{W}$ in a canonical way, by attaching a *cylindrical end*, a certain subset of the symplectisation $\mathbb{R} \times \partial W$, to W along the boundary. This is called the *completion* of W (as the Liouville vector field on $\widehat{W}$ is complete.)

Lemma/Definition 2.61 ([CE12, p. 239])**.** *Let (W, ω, X) be a connected Liouville domain.*[5] *For $\epsilon > 0$ sufficiently small, the map $\phi\colon (-\epsilon, 0] \times \partial W \to W, (t, x) \mapsto \phi_t(x)$ defined using the flow (ϕ_t) of X is an embedding to its image. For $\lambda := \iota_X \omega$ we have $\phi^*\lambda = e^r \alpha$, where $\alpha := \lambda|_{\partial W}$. Define*

$$\widehat{W} := W \cup_{\partial W} [0, \infty) \times \partial W := W \sqcup (-\epsilon, \infty) \times \partial W / \sim_\phi,$$

where we identity two points of they are mapped to each other via ϕ. Then $\widehat{W}$ is a smooth manifold (since ϕ is smooth). We extend the Liouville form λ to $\widehat{W}$ via

$$\widehat{\lambda} := \begin{cases} \lambda & \text{on } W \\ e^r \alpha & \text{on } (-\epsilon, \infty) \times \partial W. \end{cases}$$

Then $(\widehat{W}, \widehat{\lambda})$ is an exact symplectic manifold. Extending X by

$$\widehat{X} := \begin{cases} X & \text{on } W \\ \frac{\partial}{\partial_t} & \text{on } (-\epsilon, \infty) \times \partial W \end{cases}$$

yields a complete Liouville vector field on $\widehat{W}$. As a result, we obtain a Liouville manifold $(\widehat{W}, \widehat{\omega} = d\widehat{\lambda}, X)$, called the completion *of (W, ω, X).* □

Note that this definition involves no choices, since our definition of Liouville domain entails a choice of the primitive λ (or equivalently, the Liouville vector field X). On the cylindrical end, we use the contact form determined by λ and X.

Yet, we can also perform an analogous construction with any other contact form on the cylindrical end; this will be important in Section 3.3. For any other contact form $\alpha' = e^f \alpha$ on ∂W, we attach the collar neighbourhood to the cylinder along the hypersurface $\{(t, p)\colon f(p) = t\}$, i.e. we use an equivalence

$$\sim_G\colon (-\epsilon, 0] \times \partial W \to U, (t, x) \mapsto \phi_{t+f(p)}(x),$$

where U is a neighbourhood of that hypersurface. This yields the following.

[5] We restrict to connected domains for simplicity; the construction applies to non-connected manifolds just as well.

Remark 2.62. For any other contact form $\alpha' = e^f \alpha$, the completion $(\widehat{W}, \widehat{\omega})$ contains some cylindrical end $([T,\infty) \times \partial W, d(e^r\alpha')$ for some $T > 0$ for this contact form α'.

One may wonder about the converse question: is a given Liouville manifold V the completion of some Liouville domain? The answer is affirmative if and only if V is of *finite type*. We will only approximately explain what this means: for a class of Liouville manifolds called Weinstein manifolds (see Definition 2.75), being of finite type is easy to define, and there are extremely few known examples of Liouville manifolds which are not Weinstein. See Remark 2.81 for details.

Proposition 2.63 ([CE12, p. 239])**.** *The completion of a Liouville domain is a Liouville manifold of finite type. Any finite type Liouville manifold is the completion of some Liouville domain.* □

Now, we come to the definition of a Liouville filling. In the literature, the term *exact filling* is often used synonymously.

Definition 2.64. *A* Liouville filling *of a contact manifold* $(M, \xi = \ker\alpha)$ *is a Liouville domain* (W, λ) *such that* $(\partial W, \ker\lambda|_{\partial W})$ *is contactomorphic to* $(M, \ker\alpha)$*.*

Example. Any Liouville domain (W, λ) is a Liouville filling for its boundary $(\partial W, \ker\lambda|_{\partial W})$.

The natural notion of equivalence of Liouville domains or manifolds is called *Liouville homotopy*. For Liouville domains, the definition is the natural adaption of symplectic deformations.

Definition 2.65. *A* homotopy of Liouville cobordisms *is a smooth family of Liouville cobordisms* $(W, \omega_s, X_s)_{s\in[0,1]}$*, i.e. the data* ω_s *and* X_s *vary smoothly in* s*.*

One can make the same definition for Liouville manifolds. However, it is not obvious whether that yields an equivalence relation. Hence, the definition is modified to make this true by definition.

Definition 2.66. *A smooth family* $(V, \omega_s, X_s)_{s\in[0,1]}$ *of Liouville manifolds is called a* simple Liouville homotopy *if and only if there exists a smooth family of exhaustions* $V = \bigcup_{k=1}^{\infty} V_s^k$ *by compact domains* $V_s^k \subset V$ *with smooth boundaries along which* X_s *is outward pointing. A smooth family* $(V, \omega_s, X_s)_{s\in[0,1]}$ *of Liouville manifolds is called a* Liouville homotopy *if and only if it is the composition of finitely many simple Liouville homotopies.*

A natural question is whether there are Liouville manifolds which are Liouville homotopic, but without a simple Liouville homotopy (i.e., whether the concept of non-simple Liouville homotopies is necessary at all). In the special case of Weinstein homotopies (see Definition 2.88), there is such a counterexample (see Example 2.91). It is not clear to the author whether this is also a counterexample for Liouville homotopies: a priori, there could be a simple Liouville homotopy between the manifolds which is not a Weinstein homotopy.

Liouville homotopies are compatible with the completion of Liouville domains.

Lemma 2.67 ([CE12, Lemma 11.6]). *A Liouville homotopy between two Liouville domains induces a Liouville homotopy between their completions.* □

Remark 2.68. Such homotopy will be of finite type, i.e. going through finite type Liouville manifolds. It seems to be an open question (see e.g. [CE12, Example 11.7]) whether the converse is true, i.e. whether a finite type Liouville homotopy between the completions of two Liouville domains implies the existence of a Liouville homotopy between the domains themselves.

For Liouville domains, there is another notion of equivalence, called *Liouville isomorphism.*

Definition 2.69 ([Sei08, p. 3]). *A* Liouville isomorphism *between two Liouville domains* (W, λ, X) *and* (W', λ', X') *is a diffeomorphism* $\phi\colon \widehat{W} \to \widehat{W'}$ *between their completions such that* $\phi^*\widehat{\lambda'} = \widehat{\lambda} + df$*, for some compactly supported smooth function* $f\colon \widehat{W} \to \mathbb{R}$*. In particular, a Liouville isomorphism is an exact symplectomorphism between the completions* $\widehat{W}$ *and* $\widehat{W'}$*.*

A key property of Liouville homotopies is the following.

Proposition 2.70 ([CE12, Proposition 11.8]). *If two Liouville domains are Liouville homotopic, they are Liouville isomorphic. More generally, if two Liouville manifolds* (V, ω_1, X_1) *and* (V, ω_2, X_2) *are Liouville homotopic, they are exact symplectomorphic.* □

2.3. Weinstein manifolds and domains

At several points, we also need to talk about Weinstein manifolds and Weinstein domains. These are Liouville manifolds respectively domains with additional structure. This further structure is rooted in *Morse theory*, hence we briefly recall what the reader needs to know. See Milnor's book [Mil63] for a classical and the first part of Audin-Damian's textbook [AD14] for a modern account of Morse theory. Further details and background about this section can be found in Cieliebak and Eliashberg's book [CE12].

Recall. Let M be a smooth manifold. A critical point p of a smooth function $f\colon M \to \mathbb{R}$ is called *non-degenerate* if and only if the Hessian $\mathrm{Hess}_p(f)$ of f at the point p is non-degenerate. A *Morse function* on M is a smooth function $f\colon M \to \mathbb{R}$ with the property that every critical point of f is non-degenerate. The *Morse index* $\operatorname{ind} p$ of a critical point p is the maximal dimension of a linear subspace on which $\mathrm{Hess}_p(f)$ is negative definite. In particular, a local maximum of a Morse function has Morse index $\dim M$ and a local minimum has Morse index 0. If $f\colon M \to \mathbb{R}$ is Morse, the set $\mathrm{Crit}(f)$ of critical points of f is a discrete subset of M. In particular, if M is compact, $\mathrm{Crit}(f)$ is finite.

Definition 2.71. *Let M be a smooth manifold, $f\colon M \to \mathbb{R}$ a smooth function and X a smooth vector field on M. The vector field X is called* gradient-like *for f if and only if the inequality*

$$X(\phi) \geq \delta(|X|^2 + |df|^2)$$

holds, where $|X|$ is the norm with respect to some Riemannian metric on M, $|df|$ is the dual norm and δ is some positive continuous function.[6] *The pair (X, f) is called a* Lyapunov pair*.*

Observation 2.72. If X and f are a vector field resp. a smooth function on a smooth manifold M such that X is gradient-like for f, the vector field X vanishes precisely at the critical points of f, and $X(f) > 0$ holds on $M \setminus \operatorname{Crit} f$.

Example 2.73. If (M, g) is a Riemannian manifold and f any smooth function, the gradient ∇f of f with respect to the metric g is gradient-like for f.

Definition 2.74. *A smooth function $f\colon M \to \mathbb{R}$ on a smooth manifold M is called* exhausting *if and only if f is proper and bounded below. In particular, all sublevel sets $f^{-1}((-\infty, a])$ are compact.*

Example. If M is a compact manifold, any smooth function $f\colon M \to \mathbb{R}$ is exhausting.

Definition 2.75. *A* Weinstein manifold *(V, ω, X, ϕ) is a symplectic manifold (V, ω) with a complete Liouville vector field X which is gradient-like for an exhausting Morse function $\phi\colon V \to \mathbb{R}$. A* Weinstein cobordism *(W, ω, X, ϕ) is a Liouville cobordism (W, ω, X) whose Liouville vector field X is gradient-like for a Morse function $\phi\colon W \to \mathbb{R}$ which is locally constant on each boundary $\partial_\pm W$. In particular, since X is transverse to ∂W, all critical points of ϕ must lie in the interior. In both cases, the triple (ω, X, ϕ) is called a* Weinstein structure *on V resp. W. A Weinstein cobordism with $\partial_- W = \emptyset$ is called a* Weinstein domain*.*

Remark 2.76. For a Weinstein cobordism (W, ω, X, ϕ), the function ϕ is automatically exhausting since W is compact.

Example 2.77 ([CE12, Example 11.12(i)])**.** $\mathbb{C}^n$ carries the Weinstein structure

$$\omega_{\text{std}} = \sum_{i=1}^{n} dx_i \wedge dy_i, \qquad X_{\text{std}} = \frac{1}{2}\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}, \qquad \phi_{\text{std}} = \frac{1}{4}\sum_{j=1}^{n}(x_j^2 + y_j^2),$$

with coordinates $(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{C}^n$, called the *standard Weinstein structure.*

The definition of Weinstein domains (and similarly, cobordisms) has a somewhat subtle issue: the boundary ∂W of a Weinstein domain need not be connected. This is why ϕ is only required to be *locally* constant. Since all resources we consulted overlooked this issue and in addition had distinct definitions, we decide to elaborate on this further. See Section A.2 in the appendix for details.

[6] Since δ may be chosen depending on the metric, one can show that this definition is independent of the choice of Riemannian metric.

Observation 2.78. Any Weinstein manifold (V, ω, X, ϕ) can be exhausted by Weinstein domains $W_k = \{\phi \leq d_k\}$, where (d_k) is a sequence of regular values of ϕ which is strictly increasing and diverging.

The following condition will be important in the proof of the main theorem.

Definition 2.79. *A Weinstein manifold (V, ω, X, ϕ) is called of* finite type *if and only if ϕ has finitely many critical points.*

Observation 2.80. By attaching a cylindrical end, any Weinstein domain (W, ω, X, ϕ) can be completed to a finite type Weinstein manifold, called its *completion*. The details work the same way as for Liouville domains and manifolds.

Remark 2.81. Any Weinstein domain is also a Liouville domain and any Weinstein manifold (V, ω, X, ϕ) is also a Liouville manifold (V, ω, X), simply by forgetting the function ϕ. However, there are Liouville manifolds which are not even diffeomorphic to a Weinstein manifold. The first example goes back to McDuff [McD91, Theorem 1.1] (their manifold has disconnected boundary, hence is not Weinstein, by Proposition A.2). There have been further examples by Mitsumatsu [Mit95], Geiges [Gei94; Gei95] and Massot, Niederkrüger and Wendl [MNW13]. Still, the currently known examples are few and far between, and it is not well-understood what kinds of examples can occur.

Observation 2.82. In a Weinstein manifold (V, ω, X, ϕ), by Lemma 2.56 any regular level set $\Sigma := \phi^{-1}(c)$ carries a canonical contact structure ξ defined by the contact form $\alpha := (\iota_X \omega)|_\Sigma$. In particular, this applies to the boundary of a Weinstein domain.

Definition 2.83. *A* Weinstein filling *of a contact manifold $(M, \xi = \ker \alpha)$ is a Weinstein domain (W, ω, X, ϕ) such that the contact boundary $(\partial W, \ker(\iota_X \omega)|_{\partial W})$ is contactomorphic to $(M, \xi = \ker \alpha)$. In other words, a Weinstein filling of $(M, \ker \alpha)$ is a Weinstein domain which is a Liouville filling of $(M, \xi = \ker \alpha)$.*

The natural equivalence relation between Weinstein domains is a *Weinstein homotopy*. As it turns out, the correct definition is a bit subtle. In light of Definition 2.65, a first guess at a definition would be something like the following.

Wrong definition. A Weinstein homotopy on a cobordism or manifold[7] V is a smooth family $(\omega_s, X_s, \phi_s)_{s \in [0,1]}$ of Weinstein structures on V. In other words, we have smooth 1-parameter families of symplectic forms ω_s, Liouville vector fields X_s and Morse functions ϕ_s which for each $s \in [0, 1]$ define a Weinstein structure on V.

Of course, this is perfectly well-defined, but it is the *wrong* definition to make. Since the conditions on ω_s and X_s are the same as in Definition 2.65, the issue must be related to the function ϕ_s—and indeed it is: the condition that two functions be homotopic through a family of Morse functions is too restrictive. In fact, one can show the following.

[7]Since the conditions in Definition 2.66 are different from Definition 2.65, this definition should (and in the end, will) be modified accordingly. We merely want to stress that the subtle issue with this definition pertains to both Weinstein manifolds and domains.

Proposition 2.84. *Let M be a smooth manifold and $(f_t)_{t\in[0,1]}$ be a smooth family of Morse functions $f_t\colon M\to\mathbb{R}$. Then the profile*

$$C(f_t) := \{(t,s)\in[0,1]\times\mathbb{R}\colon s\in f_t(\mathrm{Crit}(f_t))\}$$

is a collection of graphs of smooth functions. More precisely, there is a family $(g_i)_{i\in I}$ of smooth functions $g_i\colon[0,1]\to M$ such that $g_i(t)\in\mathrm{Crit}(f_t)$ and $\mathrm{Crit}\, f_t=\{g_i(t)\colon i\in I\}$ for all $t\in[0,1]$. Here, I is a potentially infinite index set; we could choose $I=\mathrm{Crit}\, f_0$. $\square$

In intuitive terms, the non-degeneracy of a critical point is stable under small perturbations, so that the number of critical points must be preserved under Morse homotopies. A formal proof uses the implicit function theorem and non-degeneracy of the critical points; the details are skipped since they are not the focus of this thesis.

In general, a manifold admits Morse functions with various numbers of critical points, hence restricting to a homotopy of Morse functions is too strong. One has to slightly relax the definition, but then obtains a result of the kind we are looking for.

Definition 2.85. *A critical point p of a smooth function $f\colon M\to\mathbb{R}$ is called* embryonic *if and only if $\ker\mathrm{Hess}_p f$ is 1-dimensional and the third derivative of f in direction $\ker\mathrm{Hess}_p f$ is non-zero. See [CE12, Section 9.1] for details.*
A generalised Morse function *on a smooth manifold M is a smooth function $f\colon M\to\mathbb{R}$ such that all critical points of M are either non-degenerate or embryonic.*

Proposition 2.86 (Folklore, see e.g. [CE12, Theorem 9.4(d)])**.** *Any homotopy between two Morse functions can be perturbed to a homotopy of generalised Morse functions.* $\square$

When considering a family of generalised Morse functions, we still want the embryonic critical points to satisfy a non-degeneracy condition. This is expressed in the following definition.

Definition 2.87. *A 1-parameter family of smooth functions $f_t\colon M\to\mathbb{R}$ has a* birth-death type *critical point $p\in M$ at $t=0$ if and only if p is an embryonic critical point of f_0 and $(0,p)$ is a non-degenerate critical point of the function $(t,x)\mapsto f_t(x)$.*

Hence, the proper definition for Weinstein homotopies is to consider homotopies through generalised Morse functions, which will be Morse functions at almost all times, except that at finitely many times and critical points, a so-called *birth-death bifurcation* will occur. Let us present the details.

Definition 2.88. *A* Weinstein homotopy *on a cobordism or manifold is a smooth family of Weinstein structures $(\omega_t,X_t,\phi_t)_{t\in[0,1]}$, where we allow birth-death type degenerations, such that the associated Liouville structures (ω_t,X_t) form a Liouville homotopy.*

By the qualification "where we allow birth-death type degenerations", we mean that the functions ϕ_t are Morse except for finitely many $t \in (0,1)$ at which ϕ_t is generalised Morse and a birth-death type critical point (see Definition 2.87) occurs.

Recall that for a manifold V, homotopies of Liouville structures had a condition on the existence of a smooth family of exhaustions (see Definition 2.66), which prevents critical points from escaping to infinity. It is useful to rephrase this using the functions ϕ_t.

Definition 2.89. *Let $(X_t, \phi_t)_{t\in[0,1]}$ be a smooth family of Lyapunov pairs on a manifold V such that each ϕ_t is exhausting and Morse, except for finitely many $t \in (0,1)$ at which a birth-death type critical point occurs instead. We call (X_t, ϕ_t) a* simple Smale homotopy *if there is a family of smooth functions $c_1 \leq c_2 \leq \dots$ on $[0,1]$ such that each $c_i(t)$ for $t \in [0,1]$ is a regular value for ϕ_t and $\bigcup_k \{\phi_t \leq c_k(t)\} = V$. A* Smale homotopy *is a composition of finitely many simple Smale homotopies.*

Then, a Weinstein homotopy on V is a family of Weinstein structures $(V, \omega_t, X_t, \phi_t)_{t\in[0,1]}$, again allowing birth-death type degenerations, such that the associated Lyapunov pairs (X_t, ϕ_t) form a Smale homotopy.

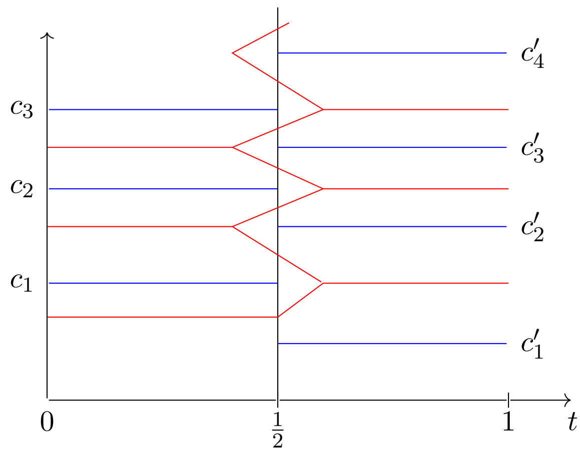
Since a Weinstein homotopy induces a Liouville homotopy, Proposition 2.70 implies that Weinstein homotopic manifolds are exact symplectomorphic.

Corollary 2.90. *If two Weinstein manifolds $(W, \omega_1, X_1, \phi_1)$ and $(W, \omega_2, X_2, \phi_2)$ are Weinstein homotopic, they are exact symplectomorphic.* □

Finally, we come to the announced example that not every Weinstein homotopy is simple.

Example 2.91. The composition of two simple Smale homotopies need not be simple!

On the right, we have the profile for a composition of two simple Smale homotopies which is not simple: the sublevel sets $\{\phi \leq c_i\}$ respectively $\{\phi \leq c_i'\}$ provide exhaustions for the restrictions of the homotopy to the intervals $[0, \frac{1}{2}]$ resp. $[1/2, 1]$. The level set $\{\phi = c_i\}$ and $\{\phi = c_i'\}$ are drawn in blue, the exhaustions are drawn in red. Is is straightforward, though a bit tedious, to check that no such exhaustion can exist over the whole interval $[0,1]$. Picture reproduced from Figure 11.1 in [CE12].



Corollary 2.92. *The composition of two simple Weinstein homotopies need not be simple.*

2.4. Loose Legendrians

As promised in the introduction, this thesis will feature an instance of the h-principle for flexible Weinstein domains. Behind this is another h-principle for so-called loose Legendrian submanifolds, which we will explain now. We already saw the definition of Legendrian submanifolds (usually called *Legendrians* for short) in Definition 2.32. In order to discuss them and their h-principle, we need to shift our point of view: isotropic submanifolds are embedded, and this embedding is important.

Definition 2.93. *Let $(M, \xi = \ker \alpha)$ be a $(2n-1)$-dimensional contact manifold and Λ a smooth manifold. An immersion $\phi\colon \Lambda \to M$ is called* isotropic *if and only if it is tangent to ξ, i.e.* $\operatorname{im} d\phi_x \subset \xi_{\phi(x)}$ *for all $x \in \Lambda$. Isotropic immersions of the maximal dimension $n-1$ are called* Legendrian*. An isotropic immersion that is not Legendrian is called* subcritical.

Observation 2.94. The image of an isotropic resp. Legendrian immersion is an isotropic resp. Legendrian immersed manifold (in the sense of Definition 2.32) of M; the image of an isotropic resp. Legendrian embedding is an embedded isotropic resp. Legendrian submanifold of M. In particular, an isotropic immersion $\phi\colon \Lambda \to M$ can only exist if $\dim \Lambda \leq n-1$.

Proof. If $\phi\colon \Lambda \to (M, \xi = \ker \alpha)$ is an isotropic immersion, for all $x \in \Lambda$ we have $d\alpha|_{d\phi(T_x\Lambda)} = d(\alpha|_{\phi(\Lambda)})(x) = 0$. Hence, each $d\phi(T_x\Lambda)$ is an isotropic subspace of the symplectic vector space $(\xi_x, d\alpha)$. The last sentence follows from Proposition 2.33. $\square$

For defining an h-principle, we need to weaken Definition 2.93 to a suitable topological statement. It turns out that following notion of a *formal* isotropic immersion/embedding is the correct one.

Definition 2.95. *Let $f\colon M \to N$ be a smooth map and $F\colon TM \to TN$ be a map on the tangent bundles. We say that F* covers *f if and only if F and f commute with the canonical projections of the tangent bundles TM and TN, i.e. if and only if the diagram on the right commutes.*

$$\begin{tikzcd} T\Lambda \arrow[r, "F"] \arrow[d] & TM \arrow[d] \\ \Lambda \arrow[r, "f"] & M \end{tikzcd}$$

Definition 2.96. *Let (M, ξ) be a $(2n-1)$-dimensional contact manifold and Λ be a smooth manifold of dimension $k \leq n-1$. A* formal isotropic embedding *of Λ into (M, ξ) is a pair (f, F^s), where $f\colon \Lambda \to M$ is a smooth embedding and $(F^s : T\Lambda \to TM)_{s\in[0,1]}$ is a smooth family of monomorphisms covering f, starting at $F^0 = df$ and ending at an isotropic monomorphism $F^1\colon T\Lambda \to \xi$. If $k = n-1$, we call this a* formal Legendrian embedding.

Our back is covered: any (genuine) isotropic embedding f can be viewed as a formal isotropic embedding $(f, F^s \equiv df)$.

The h-principle for loose Legendrians will involve isotopies of isotropic embeddings and their formal counterparts, hence we define those next. Let (M, ξ) be a $(2n-1)$-dimensional contact manifold and Λ a manifold of dimension $k \leq n-1$.

Definition 2.97. *An* isotropic isotopy *between two isotropic embeddings $f_0, f_1\colon \Lambda \to (M, \xi)$ is an isotopy $(f_t)_{t\in[0,1]}$ from f_0 to f_1 such that each embedding (f_t) is isotropic. If Λ has the maximal dimension $n-1$, we call this a* Legendrian isotopy *(since it is an isotopy through Legendrian embeddings). Two isotropic resp. Legendrians embeddings are called* isotropically isotopic *resp.* Legendrian isotopic *if and only if there is an isotropic resp. Legendrian isotopy between them.*

This definition was straightforward. Its formal analogue is as straightforward from the outside (but becomes more messy when expanding all details).

Definition 2.98. *Two isotopic embeddings $f_0, f_1\colon \Lambda \to (M, \xi)$ are called* formally isotropically isotopic *if and only if they are isotopic as formal isotropic embeddings, i.e. there exists a smooth 1-parameter family $(f_t, F_t^s)_{t\in[0,1]}$ of formal isotropic embeddings connecting (f_0, F_0^s) and (f_1, F_1^s).*

Expanding this, a formal isotropic isotopy between two formal isotropic embeddings (f_0, F_0^s) and (f_1, F_1^s) consists of an isotopy $(f_t\colon \Lambda \to M)_{t\in[0,1]}$ between f_0 and f_1 and a family $F_t^s\colon T\Lambda \to TM$ of monomorphisms, such that each F_t^s covers f_t, one has $(F_t^s)_{t=0} \equiv F_0^s$ and $(F_t^s)_{t=1} \equiv F_1^s$ and, for all t, $F_t^0 = df_t$ and $F_t^1\colon T\Lambda \to \xi$ is isotropic.

It is apparent that any isotopic isotropic embeddings are necessarily formally isotopic. The h-principle states that under certain circumstances, the converse result also holds. For subcritical isotropic manifolds, this is much easier to prove and was already shown by Gromov.

Theorem 2.99 (Gromov, [EM02; Gro86])**.** *Let (M, ξ) be a $(2n-1)$-dimensional contact manifold and Λ a smooth manifold of dimension $k < n-1$.*[8]

(a) For any formal isotropic embedding $(f\colon \Lambda \to M, F^s)$, there is a genuine isotropic embedding $g\colon \Lambda \to M$ such that $(f\colon \Lambda \to M, F^s)$ and (g, dg) are formally isotropically isotopic.

(b) Suppose two isotropic embeddings $f_0, f_1\colon \Lambda \to M$ are connected by a formal isotropic isotopy $(f_t, F_t^s)_{s,t\in[0,1]}$. Then, there exists a genuine isotropic isotopy (g_t) connecting $g_0 = f_0$ and $g_1 = f_1$ which is homotopic to the formal isotropic isotopy (f_t, F_t^s) through formal isotropic isotopies. □

[8] These statements also hold for smooth families in several parameters, e.g. parametrised by $[0,1]^n$ for any n; the conceptual statement behind this is that the natural inclusion

$$\{\text{isotropic embeddings } f\colon \Lambda \to M\} \hookrightarrow \{\text{formal isotropic embeddings } \Lambda \to M\}$$

is a weak homotopy equivalence. There are also relative versions of the h-principle, i.e. for (formal) embeddings with prescribed values in a neighbourhood of a given closed subset.

For Legendrian submanifolds, this is not always true. However, in dimension three (so that Legendrians are 1-dimensional), there is an h-principle for Legendrians in so-called *overtwisted* contact manifolds [Dym01; EF09]. More strikingly, in *any* contact manifold of dimension at least five, there is a class of so-called *loose* Legendrians which satisfy an h-principle. A Legendrian is defined to be loose if and only if there is a chart around each point in which the Legendrian has a precise standard form; such a chart is called a *loose chart*. We omit the details since they are technical and not very enlightening as to why the h-principle is true; the following statement is all we need to know for the purpose of this thesis. In the theorem, a *knot* denotes a parametrised embedding of a connected manifold, hence a Legendrian knot is a connected embedded Legendrian submanifold.

Theorem 2.100 (Murphy, [Mur19, Theorem 1.2, 1.3])**.** *Let (M, ξ) be a contact manifold of dimension $2n - 1 \geq 5$ and Λ be a $(n-1)$-dimensional smooth manifold.*

(a) Any formal Legendrian embedding $(f: \Lambda \hookrightarrow M, F^s: T\Lambda \to TM)$ can be C^0-approximated by a loose Legendrian embedding $\tilde{f}: \Lambda \to M$ which is formally Legendrian isotopic to (f, F^s).

(b) Any smooth isotopy $(f_t: \Lambda \to M)_{t\in[0,1]}$ which begins with a loose Legendrian embedding f_0 can be C^0-approximated by a Legendrian isotopy starting at f_0.

(c) Let $(f_t, F_t^s)_{s,t\in[0,1]}$ be a formal Legendrian isotopy connecting two loose Legendrians knots f_0 and f_1. There exists a Legendrian isotopy $\tilde{f}_t$ connecting $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$ which is C^0-close to f_t and is homotopic to the formal isotopy (f_t, F_t^s) through formal isotopies with fixed endpoints. □

For completeness, let us also mention the following result, which underscores that the notion of loose Legendrians is non-trivial.

Proposition 2.101 ([Che02; EES05])**.** *For any $n \geq 1$, there are pairs of Legendrian knots in $\mathbb{R}^{2n+1}$ which are formally but not genuinely Legendrian isotopic.* □

2.5. Surgery and handle attachment

In this section, we explain the closely related operations of *surgery* and *handle attachment*. Handle attachment is needed to state the *handlebody decomposition* (Proposition 2.125) of Weinstein domains, which provides some underpinning for a key step in Zhou's proof. In Section 5.2 we will see that *flexible* Weinstein domains (see Definition 2.131 below) and asymptotically dynamically convex contact manifolds (Definition 5.8) behave nicely under boundary connected sums (which is a particular form of handle attachment).

Both surgery and handle attachment can be performed on a purely topological level, but are also compatible with additional structure. *Weinstein handle attachment* involves attaching a handle to a Weinstein domain and extending the Weinstein structure to the result. If M is a contact manifold, attaching a handle to a trivial symplectic cobordism (Definition 2.110 below) $[-1,1]\times M$ yields a cobordism between M and the manifold M' obtained by performing surgery on M. Hence, M' is also a contact manifold; we say M' is obtained from M via *contact surgery*. The foundational material in this section can be found in Geiges' book [Gei08, Chapter 6]. For some details about Weinstein domains, Weinstein's original paper [Wei91] is helpful.

Notation. As in the whole thesis, we use $\mathbb{D}^{k+1}\subset\mathbb{R}^{k+1}$ to denote the $(k+1)$-dimensional closed unit disc and $\mathbb{S}^k=\partial\mathbb{D}^{k+1}$ for its boundary, the k-dimensional unit sphere.

We begin with presenting the constructions on the level of smooth[9] manifolds, and we explain the idea of surgery first. The idea behind surgery is that the product manifolds $\mathbb{S}^k\times\mathbb{D}^{n-k}$ and $\mathbb{D}^{k+1}\times\mathbb{S}^{n-k-1}$ both have the manifold $\mathbb{S}^k\times\mathbb{S}^{n-k-1}$ as boundary. Hence, one can replace an embedded copy of $\mathbb{S}^k\times\mathbb{D}^{n-k}$ by a copy of $\mathbb{D}^{k+1}\times\mathbb{S}^{n-k-1}$ (in an appropriate way to preserve the smooth manifold structure). This replacement depends on the orientation of the embedded discs and spheres, which is given by a choice of *framing*.

Definition 2.102. *A* framing *of an embedded submanifold $L\subset M$ is a choice of trivialisation of the normal bundle $N_M(L)$.*

The following statement follows from the tubular neighbourhood theorem.

Lemma 2.103. *Let M be an oriented n-dimensional manifold and $L\subset M$ be an oriented k-dimensional submanifold. Each trivialisation of the normal bundle $N_M(L)$ induces an embedding $L\times\mathbb{D}^{n-k}\to M$.* □

Convention. In this section, we only consider orientable manifolds.

Lemma/Definition 2.104 ([Gei08, Definition 6.1.1])**.** *Let M be an n-dimensional manifold. Given an embedding $\phi\colon\mathbb{S}^k\times\mathbb{D}^{n-k}\to M$, the space*

$$M' := (M\setminus\mathbb{S}^k\times\operatorname{Int}(\mathbb{D}^{n-k}))\cup_{\mathbb{S}^k\times\mathbb{S}^{n-k-1}}(\mathbb{D}^{k+1}\times\mathbb{S}^{n-k+1})$$

formed by identifying both factors along

$$\mathbb{S}^k\times\mathbb{S}^{n-k-1}=\partial(M\setminus(\mathbb{S}^k\times\operatorname{Int}(\mathbb{D}^{n-k})))=\partial(\mathbb{D}^{k+1}\times\mathbb{S}^{n-k+1}).$$

is an n-dimensional manifold. One says that M' is obtained from M by performing surgery along *$\mathbb{S}^k\subset M$.* □

Remark 2.105. The definition of M' incurred various choices: for the embedding ϕ as well as for the identification. As it turns out, M' is determined up to diffeomorphism by the isotopy class of the embedding $\mathbb{S}^k\times\mathbb{D}^{n-k}\to M$: if $\phi,\tilde{\phi}\colon\mathbb{S}^k\times\mathbb{D}^{n-k}\to M$ are isotopic, the resulting manifolds M' and $\widetilde{M}'$ are diffeomorphic [Gei08, Remark 6.1.2].

[9]Unless stated otherwise, all manifolds, cobordisms and maps considered in this section are smooth.

Example 2.106. Suppose $M = M_1 \sqcup M_2$ is the disjoint union of two connected components and choose two points $x \in M_1$, $y \in M_2$. This yields an embedding $\mathbb{S}^0 \to M$. Performing surgery on M along $\mathbb{S}^0$ yields the *connected sum* $M_1 \# M_2$; equivalently, we could take two discs (with opposite orientations) in the M_i and connect their boundaries by a tube.

An important remark for later considerations is that surgery is reversible.

Observation 2.107 ([Gei08]). If M' is obtained from M by surgery along $\mathbb{S}^k$, performing surgery on M' along the sphere $\mathbb{S}^{n-k-1}$ yields M again. □

As indicated already, surgery can also be described via *handle attachment*. Let M be an oriented n-dimensional manifold and consider the cylinder $[-1,1] \times M$. Given an embedding $\mathbb{S}^k \times \mathbb{D}^{n-k} \to M \equiv \{1\} \times M$, we form the n-dimensional manifold

$$W = ([-1,1] \times M) \cup_{\mathbb{S}^k \times \mathbb{D}^{n-k}} (\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}).$$

We say that W is obtained from $[-1,1] \times M$ by *attaching a* $(k+1)$*-handle* or *handle of index* $k+1$ to the boundary component $\{1\} \times M$; we call $\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}$ a $(k+1)$*-handle*.

This definition is not fully rigorous: W is not a smooth manifold since it has *corners*[10] at $\mathbb{S}^k \times \mathbb{S}^{n-k-1}$. One can correct this by "smoothing the corners"; we will not explain this in detail, but refer to the references in [Gei08, p. 289]. With that resolved, the boundary of W is given as the disjoint union of $\overline{M}$ (i.e. the manifold M with its orientation reversed) and the result M' of performing surgery on M via $\mathbb{S}^k \times \mathbb{D}^{n-k} \subset M$. In other words, we have the following result.

Proposition 2.108 ([Gei08])**.** *If M is an n-dimensional manifold and $\phi\colon \mathbb{S}^k \times \mathbb{D}^{n-k} \to M$ an embedding, the manifold W obtained by setting*

$$W = ([-1,1] \times M) \cup_{\mathbb{S}^k \times \mathbb{D}^{n-k}} (\mathbb{D}^{k+1} \times \mathbb{D}^{n-k})$$

and then smoothing the corners is a smooth cobordism between M and M', where M' is the result of performing surgery on M with ϕ. □

There is an alternative description for handle attachment which doesn't require smoothing of corners and makes it much easier to carry over additional structure. For $1 \le k \le n$, we model a $(k+1)$-handle not by $\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}$, but instead by the subset

$$H := \{(x,y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k} \colon -1 \le |y|^2 - |x|^2 \le 1 \text{ and } |x| \cdot |y| < \sinh 1 \cdot \cosh 1\}.$$

The manifold H is not diffeomorphic to $\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}$, but rather a copy of $\mathbb{D}^{k+1} \times \mathbb{D}^{n-k}$ with the corners cut off. We call the subset

$$\partial^- H := \{(x,y) \in H \colon |y|^2 - |x|^2 = -1\}$$

[10] See e.g. [Lee02, Chapter 14] for a precise definition of manifolds with corners.

the *lower boundary* of H; the *upper boundary* of H is the set

$$\partial^+ H := \{(x,y) \in H \colon |y|^2 - |x|^2 = 1\}.$$

The $(k+1)$-disc $\mathbb{D}^{k+1} \times \{0\} \subset H$ is called the *core* of the handle H, the $(n-k)$-disc $\{0\} \times \mathbb{D}^{n-k}$ is called the *belt disc* or *co-core*; the $(n-k-1)$-sphere $\{0\} \times \partial\mathbb{D}^{n-k} \subset \partial H$ is called the *belt sphere*.

To inspire further confidence in our model handle, note that the lower boundary is diffeomorphic to $\mathbb{S}^k \times \mathrm{Int}(\mathbb{D}^{n-k})$ via the map

$$\partial^- H \ni (u \cosh r, v \sinh r) \mapsto (u, rv), \qquad \text{where } u \in \mathbb{S}^k, v \in \mathbb{S}^{n-k-1}, 0 \leq r < 1.$$

Similarly, the upper boundary is diffeomorphic to $\mathrm{Int}(\mathbb{D}^{k+1}) \times \mathbb{S}^{n-k-1}$ via the map

$$\partial^+ H \ni (u \sinh r, v \cosh r) \mapsto (ru, v), \qquad \text{where } u \in \mathbb{S}^k, v \in \mathbb{S}^{n-k-1}, 0 \leq r < 1.$$

This is slightly imprecise since for $r = 0$, the coordinate v resp. u is not unique any more (as $\sinh(0) = 0$). However, the maps are independent of the choice made in this case (yielding $(u,0) \mapsto (u,0)$ for $\partial^- H$ and $(0,v) \mapsto (0,v)$ for $\partial^+ H$), hence still well-defined. Smoothness at $r = 0$ is straightforward to check.

We can use H to describe the cobordism W above: let M be an n-dimensional manifold and $\phi \colon \mathbb{S}^k \times \mathbb{D}^{n-k} \to M$. Define W as the quotient

$$([-1,1] \times (M \setminus \phi(\mathbb{S}^k \times \{0\})) \sqcup H)/\sim,$$

via the following identification. For a given point $(x_0, y_0) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k}$, consider the curve $\gamma \colon \mathbb{R}^+ \ni t \mapsto (t^{-1}x_0, ty_0)$. Note that $|x||y|$ is constant along γ, and that γ is an integral curve of the gradient to the function $(x,y) \mapsto |y|^2 - |x|^2$. For each $u \in \mathbb{S}^k$, $v \in \mathbb{S}^{n-k-1}$, $0 < r < 1$ and $c \in [-1,1]$ we identify $(c, \phi(u, rv)) \in [-1,1] \times (M \setminus \phi(\mathbb{S}^k \times \{0\}))$ with the unique point $(x,y) \in H$ defined by the conditions

- $|y|^2 - |x|^2 = c$ and
- (x,y) lies on the trajectory γ through the point $(u \cosh r, v \sinh r)$.

This defines a diffeomorphism

$$[-1,1] \times \phi(\mathbb{S}^k \times \mathrm{Int}(\mathbb{D}^{n-k} \setminus \{0\})) \to H \cap (\mathbb{R}^{k+1} \setminus \{0\}) \times (\mathbb{R}^{n-k} \setminus \{0\})$$

of open subsets; hence the resulting space W is a smooth $(n+1)$-dimensional manifold. The sphere $\phi(\mathbb{S}^k \times \{0\}) \subset M$ is called the *attaching sphere* for the handle H.

This construction works for general smooth manifolds. Let us now extend this construction to preserve additional symplectic or contact structure. The definition of framing needs to be adapted accordingly.

Recall (Section 2.1)**.** If $L \subset M$ is an isotropic submanifold of a contact manifold (M, ξ) and $x \in L$, we have $T_pL \subset (T_pL)^\omega$, where $(T_pL)^\omega$ is the symplectic complement of T_pL in the symplectic vector space ξ_p.

Lemma/Definition 2.109 ([Gei08, Definition 6.2.1])**.** *Let L be an isotropic submanifold of a contact manifold $(M,\ \xi = \ker\alpha)$. The quotient bundle*

$$SN_M(L) := (TL)^\omega / TL$$

is a symplectic vector bundle with symplectic structure induced by $d\alpha$. We call $SN_M(L)$ the symplectic normal bundle *of L in M.*

We begin by explaining how to perform contact surgery, which will require extending a symplectic structure across handle attachment. The following definition generalises Liouville cobordisms.

Definition 2.110 ([Gei08, Definition 5.2.1])**.** *Let $(M_\pm, \xi_\pm = \ker\alpha_\pm)$ be closed contact manifolds of dimension $2n-1$, oriented using the contact forms $\alpha_\pm$. A* symplectic cobordism *from (M_-, ξ_-) to (M_+, ξ_+) is a compact $2n$-dimensional symplectic manifold (W, ω), oriented by the volume form ω^n, with the following properties.*

- *The oriented boundary of W equals $\partial W = M_+ \sqcup \overline{M_-}$, where $\overline{M_-}$ stands for the manifold M_- with reversed orientation.*
- *In a neighbourhood of ∂W, there is a Liouville vector field X for ω which is pointing outwards along M_+ and inwards along M_-.*
- *The 1-form $\alpha := \iota_X\omega$ restricts to $TM_\pm$ as a contact form for $\xi_\pm$.*

Given an isotropic sphere with trivial normal bundle, we need to identify a suitable neighbourhood of the sphere with a corresponding neighbourhood in the model handle H. The following enables us to do so.

Theorem 2.111 ([Gei08, Theorem 6.2.2])**.** *Let $(M_1, \ker\alpha_1)$ and $(M_2, \ker\alpha_2)$ be contact manifolds with closed isotropic submanifolds $L_1 \subset M_1$ and $L_2 \subset M_2$. Suppose there is an isomorphism of symplectic normal bundles $\Phi\colon SN_{M_1}(L_1) \to SN_{M_2}(L_2)$ which covers a diffeomorphism $\phi\colon L_1 \to L_2$. Then the diffeomorphism ϕ extends to a contactomorphism $\psi\colon \mathcal{N}(L_1) \to \mathcal{N}(L_2)$ of suitable neighbourhoods $\mathcal{N}(L_i)$ of L_i such that $d\psi|_{SN_{M_1}(L_1)} = \Phi$.*

About the proof. The proof applies two classical results in contact geometry, namely *Darboux' theorem* for contact forms and the *Moser deformation trick*. We skip the details since they would lead us too far. □

For a contact manifold $(M, \xi = \ker\alpha)$, the trivial cobordism $[-1,1] \times M$ has a symplectic structure via $\omega_{(r,p)} = d(e^r\alpha_p)$; the cobordism is a subset of the symplectisation $\mathbb{R} \times M$ of M. Next, we describe the handle to attach to this cobordism. We consider the handle corresponding to an isotropic $(k-1)$-sphere L in a $(2n-1)$-dimensional contact manifold M, with $1 \leq k \leq n$.

The situation is similar to the topological surgery before, except that we work with coordinates $x = (q_1, \ldots, q_k) \in \mathbb{R}^k$ and $y = (q_{k+1}, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n-k}$. On $\mathbb{R}^{2n} = \mathbb{R}^k \times \mathbb{R}^{2n-k}$, we have the standard symplectic form $\omega_0 = \sum_{j=1}^n dp_j \wedge dq_j$. Instead of the gradient of the function $(x, y) \mapsto |y|^2 - |x|^2$, we consider the flow of the Liouville[11] vector field given by

$$Y := \sum_{j=1}^{k} (-q_j\, \partial_{q_j} + 2p_j\, \partial_{p_j}) + \frac{1}{2} \sum_{j=k+1}^{n} (q_j\, \partial_{q_j} + p_j\, \partial_{p_j}).$$

Note that Y is the gradient vector field of the function

$$g\colon (q, p) \mapsto \sum_{j=1}^{k} (-\frac{1}{2} q_j^2 + p_j^2) + \frac{1}{4} \sum_{j=k+1}^{n} (q_j^2 + p_j^2).$$

Let $\mathcal{N}_H \cong \mathbb{S}^{k-1} \times \operatorname{Int}(\mathbb{D}^{2n-k})$ be an open neighbourhood in the hypersurface $g^{-1}(-1) \subset \mathbb{R}^{2n}$ of the $(k-1)$-sphere

$$\mathbb{S}_H^{k-1} := \Big\{ \sum_{j=1}^{k} q_j^2 = 2,\ q_{k+1} = \ldots = q_n = p_1 = \ldots = p_n = 0 \Big\}.$$

This $\mathcal{N}_H$ will play the role of the lower boundary. We define the symplectic handle H as the set of points $(q, p) \in \mathbb{R}^{2n}$ which satisfy the inequality $-1 \leq g(q, p) \leq 1$ and lie on a gradient flow line of g through a point of $\mathcal{N}_H$. The computations above show that the handle H has a Weinstein structure (ω_0, Y, g). This generalises the standard Weinstein structure in Example 2.77.

The Liouville vector field Y is transverse to the level sets of g, hence the 1-form $\alpha_0 := \iota_Y \omega_0$ induces a contact form on the lower and upper boundary of H. With respect to this contact form, $\mathbb{S}_H^{k-1}$ is an isotropic sphere in the lower boundary.

The second step is gluing the handle H symplectomorphically to $[-1, 1] \times M$, so we obtain a symplectic structure on the cobordism defining the contact surgery. The following lemma allows us to do so.

Lemma 2.112 ([Gei08, Lemma 5.2.4])**.** *For $i = 1, 2$, let M_i be a hypersurface in a symplectic manifold (W_i, ω_i) and X_i be a Liouville vector field defined in a neighbourhood of M_i that transverse to M_i. Write j_i for the inclusion of M_i into W_i; then $\alpha_i := j_i^*(\iota_{Y_i}\omega)$ is a contact form on M_i. Given a contactomorphism $\phi\colon (M_1, \ker \alpha_1) \to (M_2, \ker \alpha_2)$, extend it to a diffeomorphism $\tilde{\phi}$ from a cylindrical neighbourhood of M_1 in W_1 to a corresponding neighbourhood of M_2 in W_2 by sending the flow lines of X_1 to X_2. Then $\tilde{\phi}$ is a symplectomorphism.* □

[11] Y is a Liouville vector for ω_0: a simple computation shows that $\alpha := \iota_Y \omega_0$ defines a primitive for ω_0.

One final issue concerns the framing of the surgery corresponding to the cobordism with the symplectic handle attached. Regarding that, we have the following.

Lemma 2.113 ([Gei08, p. 298])**.** *Let $(M, \xi = \ker\alpha)$ be a contact manifold, R be the Reeb vector field of α and J a compatible complex structure on $(\xi, d\alpha)$. For an isotropic sphere $\mathbb{S}^{k-1} \subset M$, the natural trivialisation of $\langle R\rangle \oplus J(T\mathbb{S}^{k-1})$, and any choice of symplectic trivialisation of $SN_M(\mathbb{S}^{k-1})$ determine a bundle isomorphism $\phi\colon SN_M(\mathbb{S}^{k-1}) \to SN_{\partial H}(\mathbb{S}^{k-1}_H)$.* □

We call this framing of $\mathbb{S}^{k-1}$ the *natural framing* induced by the chosen trivialisation of the symplectic normal bundle. Hence, we have outlined the proof of the following.

Theorem 2.114 ([Gei08, Theorem 6.2.5])**.** *Let $\mathbb{S}^{k-1}$ be an isotropic sphere in a contact manifold $(M, \xi = \ker\alpha)$ with a trivialisation of the symplectic normal bundle $SN_M(\mathbb{S}^{k-1})$. There is a symplectic cobordism from (M, ξ) to the manifold M' obtained from M by surgery along $\mathbb{S}^{k-1}$ using the natural framing. In particular, the manifold M' carries a contact structure that coincides with the one on M away from the surgery region.* □

A natural question is when the hypotheses of Theorem 2.114 are satisfied.

Remark 2.115. For $k = 1$, the assumptions of the theorem are trivial. Hence, one can always form the connected sum of equidimensional contact manifolds (see below).
If $\dim M = 2n-1$, the rank of $SN_M(\mathbb{S}^{k-1})$ is $2(n-k)$. Hence, contact surgery is always possible along a Legendrian sphere $\mathbb{S}^{n-1}$, and the framing for the contact surgery is completely determined by the embedding of the sphere.

Moreover, in the theorem above, the framing for a contact surgery is determined by a choice of trivialisation of the symplectic normal bundle, hence one may wonder if this trivialisation depends on the embedding $\mathbb{S}^{k-1} \subset M$, say up to isotopy. This is where another h-principle (for isotropic immersions, generalising Theorem 2.99) comes in; in the end we obtain the following.

Theorem 2.116 ([Gei08, Theorem 6.3.1])**.** *Let (M, ξ) be a $(2n-1)$-dimensional contact manifold with $n > 2$. Assume M contains a $(k-1)$-dimensional embedded sphere with trivial (topological) normal bundle, $1 \le k \le n$, i.e. there is an embedding*

$$f\colon \mathbb{S}^{k-1} \times \mathbb{D}^{2n-k} \to M.$$

Let $(W, \omega) = ([-1, 1] \times M, d(e^r\alpha))$[12]. Let $W' = W \cup_f H$ be the manifold obtained from W by attaching a k-handle H along its lower boundary $\partial^- H = \mathbb{S}^{k-1} \times \mathbb{D}^{2n-k}$ using f, and call M' the new boundary (i.e. the result of surgery of M).

[12] for the experts: (W, ω) could be any symplectic manifold which has (M, ξ) as a convex boundary component.

Let J be any compatible almost complex structure on (W, ω). If J extends over H to an almost complex structure on all of W', then W' carries a symplectic form ω' and a ω'-compatible almost complex structure homotopic to J such that M' is the convex boundary of (W, ω'). The induced contact structure ξ' on M' is the result of performing contact surgery along an isotropic sphere topologically isotopic to $f(\mathbb{S}^{k-1} \times \{0\})$. □

An important special case of the above construction are the boundary and contact connected sum.

Definition 2.117. *Let M and M' be two contact manifolds of the same dimension and consider an embedding $\mathbb{S}^0 \to M \sqcup M'$ with one point in each component. We call the result of performing contact surgery along $\mathbb{S}^0$ the* contact connected sum *$M \# M'$ of M and M'. Note that $M \# M'$ is a contact manifold, which topologically is just the connected sum of M and M'.*

Definition 2.118. *Let W and W' be two n-dimensional Liouville or Weinstein manifolds. Fix two points $p \in \partial W$ and $q \in \partial W'$, attach a 1-handle to $W \sqcup W'$ (so that the map $\{a, b\} = \mathbb{S}^0 \times \mathbb{D}^n \to \partial(W \sqcup W')$ maps $\{a\} \times \mathbb{D}^n$ to p and $\{b\} \times \mathbb{D}^n$ to q) and extend the Liouville or Weinstein structure to the handle. The resulting Liouville or Weinstein manifold is called the* boundary connected sum *$W \natural W'$ of W and W'.*

Proposition 2.119. *If W and W' are two n-dimensional manifolds with boundary M and M' respectively, we have $\partial(W \natural W') = M \# M'$: the boundary of the boundary connected sum is the contact connected sum of the boundaries. Moreover, if W and W' are Liouville or Weinstein fillings of M and M', respectively, then $W \natural W'$ is a Liouville or Weinstein filling of the contact connected sum $M \# M'$.* □

Remark 2.120. For the boundary connected sum, the attaching spheres are just points in each manifold, hence clearly isotopic to each other. (For general subcritical isotropic submanifolds, Gromov's h-principle, Theorem 2.99, shows that any two attaching spheres are smoothly isotopic.) This isotopy can be turned into a symplectic deformation by performing the isotopy via the flow of a suitable Hamiltonian vector field. Then, one has to argue that this deformation also preserves a given Liouville or Weinstein structure (i.e. yields a Liouville resp. Weinstein homotopy); for that the Moser deformation trick is useful again.

We already saw that the symplectic handle carries a natural Weinstein structure. It is possible to preserve the Weinstein structure under the handle attachment; we refer the reader to Weinstein's paper [Wei91], where Theorem 4.2 contains the details for extending the Liouville structure. In the end, the following result holds.

Proposition 2.121. *Let W be a Liouville or Weinstein cobordism, M a component of ∂W (hence M is a contact manifold) and $L \subset M$ an isotropic sphere with a trivialisation of $SN_L(M)$. Then the cobordism W' obtained from W by attaching a symplectic $(\dim L + 1)$-handle along a neighbourhood of L carries a Liouville or Weinstein structure, and W' connects M with the result M' of performing contact surgery to M along L.* □

We speak of contact surgery along a subcritical isotropic submanifold as *subcritical surgery*, and refer to contact surgery along a loose Legendrian submanifold as *flexible surgery.* As a consequence, Weinstein fillable contact manifolds are closed under subcritical and flexible surgery. In particular, the boundary connected sum of Weinstein domains is a Weinstein domain.

2.6. Weinstein handle decomposition and flexible Weinstein domains

The presence of the Morse function ϕ endows Weinstein domains with a special structure of a *handle decomposition.* The basic idea is that any Weinstein domain of dimension $2n$ can be obtained from the $2n$-dimensional standard ball $\mathbb{D}^{2n}$ by a finite or countable sequence of handle attachments. At a topological level, the existence of a handle decomposition follows from classical Morse theory, with results like the following.

Proposition 2.122 ([Mil63, Theorems 3.1, 3.2, 3.5])**.** *Let M be a smooth manifold and $\phi\colon M \to \mathbb{R}$ be an exhausting Morse function. If $a < b$ and ϕ has no critical value between a and b, then the sublevel sets $M^a := \{\phi \leq a\}$ and $M^b := \{\phi \leq b\}$ are homeomorphic.*[13] *Suppose there is a critical point p with critical value $c \in (a, b)$. Let k be the Morse index of p. Then we have $M^b \cong M^a \cup H$: the sublevel set M^b is homotopy equivalent to the set M^a with a handle H of index k attached. In general, M has the homotopy type of a CW complex with one k-cell for each critical point of Morse index k.* □

While the above is the kind of statement we are looking for, it does not involve the symplectic structure. Since we want to study Weinstein domains up to symplectomorphism, we need a stronger statement, where the handles are attached in accordance with the symplectic and Weinstein structure. To make this precise, we need one further concept from Morse theory: the *stable manifold* of a critical point.

Lemma/Definition 2.123 ([CE12, p. 190 and Lemma 9.9; AD14, p. 28])**.** *Let M be a smooth manifold, $\phi\colon M \to \mathbb{R}$ a Morse function and X a gradient-like vector field for ϕ. Denote the flow of X by X^s. Let p be a critical point of ϕ. Then the set*

$$M_p := \{x \in M \colon \lim_{s\to\infty} X^s(x) = p\}$$

is a smooth submanifold of M whose dimension equals the Morse index of p.
The submanifold M_p is called the stable manifold *of p.* □

In our case, the Morse function ϕ is defined on a Liouville domain. This implies an even stronger statement about the stable manifolds of critical points.

[13] In fact, they are (smoothly) deformation equivalent, hence diffeomorphic.

Proposition 2.124 ([CE12, Proposition 11.9(b)])**.** *Let (W, ω, X, ϕ) be a Weinstein domain and $p \in \operatorname{Crit}(\phi)$ a critical point. Then the stable manifold of p is an isotropic submanifold of W.* □

While the statement of the handle decomposition of Weinstein domains is implicit in the literature (e.g. in [CE12, p. 244]), the author knows of no place where the following precise phrasing can be found.

Proposition 2.125. *Let (W, ω, X, ϕ) be a $2n$-dimensional Weinstein domain. Let $c_1 < c_2 < \ldots < c_N$ be the critical values of ϕ. Let $\epsilon_1, \ldots, \epsilon_N > 0$ be chosen such that $c_{i-1} < c_i - \epsilon_i < c_i < c_i + \epsilon_i < c_{i+1}$ for $i = 2, \ldots, N-1$ and $c_1 + \epsilon_1 < c_2$ and $c_{N-1} < c_N - \epsilon_N$. For each c_i, let $p_{i1}, \ldots, p_{ik_i}$ be the sequence of critical points corresponding to that value. Then there is a sequence of $2n$-dimensional Weinstein domains $W_1, W_2, \ldots W_N$ such that*

- *W_1 is the disjoint union of k_0 discs $\mathbb{D}^{2n}$ with the standard Weinstein structure (see Example 2.77),*
- *each W_i is Weinstein homotopic to $\{\phi \leq c_i + \epsilon_i\} \subset W$, yielding an exact symplectomorphism $\phi_i \colon W_i \to \{\phi \leq c_i + \epsilon_i\} \subset W$,*
- *each W_i is obtained from W_{i-1} by attaching a handle of index $\operatorname{ind}(p_{ij})$ for each j to W_{i-1}, and extending the Weinstein structures to each handle. The attaching spheres are contained in the image of a sublevel set of ϕ (hence (contact) isotropic submanifolds), whereas the core discs of the handles are contained in $\phi_i(W_{p_{ij}})$, the image of the stable manifold of p_{ij}.* □

Combining Proposition 2.125 with Proposition 2.124 and Proposition 2.17, we deduce the following result.

Corollary 2.126. *A $2n$-dimensional Weinstein domain has a handlebody decomposition with handles of index at most n.*

In particular, the topology of a Weinstein domain is rather restricted.

Corollary 2.127. *Let W be a $2n$-dimensional Weinstein manifold of finite type. The singular homology $H_k(W; \mathbb{Z})$ is finitely generated, and $H_k(W; \mathbb{Z}) = 0$ for $k > n$.* □

We have now encountered the necessary concepts to introduce flexible Weinstein domains. Similar to embedded isotropic submanifolds in Section 2.4, critical points on Weinstein domains come in two kinds.

Definition 2.128. *Let (W, ω, X, ϕ) be a $2n$-dimensional Weinstein domain. A critical point p of ϕ is called* critical *if and only if it has Morse index n and* subcritical *otherwise.*

Definition 2.129. *A Weinstein domain W is called* subcritical *if and only if all its critical points are subcritical.*

Remark 2.130. A subcritical Weinstein domain has a handlebody decomposition whose attaching spheres are subcritical isotropic manifolds: the dimension of each handle attaching sphere equals the Morse index of the corresponding critical point, and by definition a $2n$-dimensional subcritical domain has no critical points of Morse index n.

We will see again that subcritical Weinstein domains satisfy an h-principle (which might not be too surprising, taking into account Remark 2.130 and their h-principle, Theorem 2.99). However, this result also extends beyond subcritical Weinstein domains. In light of Murphy's h-principle (Theorem 2.100), one could guess that loose Legendrians are involved. More precisely, the attaching sphere of each critical handle should be a *loose* Legendrian submanifold.

The correct definition of a flexible Weinstein domain is somewhat subtle. Morally, it is indeed that "each critical handle in the handlebody description is attached along a loose Legendrian", but there are two subtleties in making this precise. The first issue is that one wants to allow Weinstein manifolds of infinite type. Hence, one cannot just say "the attaching spheres of all critical handles form a loose Legendrian link", since a Legendrian link (see below) has only *finitely* many connected components.

The more fundamental issue is that flexible Weinstein domains should be invariant under Weinstein homotopy, since that is the natural equivalence for Weinstein domains. The first definition of flexibility (what we call *explicitly flexible* below) was made by Cieliebak and Eliashberg [CE12, Definition 11.28]. They did not know whether their definition was invariant under Weinstein homotopy; Murphy and Siegel showed that it was not [MS]. Hence, the correct definition of flexibility goes in two steps. Recall that a Legendrian knot is a connected Legendrian submanifold; a Legendrian *link* is the union of finitely many disjoint Legendrian knots.

Definition 2.131. *A Weinstein domain (W, ω, X, ϕ) is called* explicitly flexible *if and only if there is a sequence $c_1 < \min(\phi) < c_2 < \ldots < \max(\phi) < c_N$ of regular values of ϕ such that*

- *each Weinstein cobordism $W_i := \{c_i \leq \phi \leq c_{i+1}\}$ for $i = 1, \ldots, k-1$ contains exactly one critical point of ϕ, and*
- *in each W_i, the attaching sphere of the critical point is either subcritical or a loose Legendrian knot in $\phi^{-1}(c_i)$.*

A Weinstein domain is called flexible *if and only if it is Weinstein homotopic to an explicitly flexible domain.*

This definition adapts naturally to Weinstein manifolds (of finite or infinite type). Note that flexible Weinstein domains are invariant under Weinstein homotopies by definition.

Remark 2.132. The literature also contains an alternative definition, which Lazarev is using: a Weinstein domain is called explicitly flexible if and only if there exists an increasing sequence (c_k) of regular values of ϕ such that each Weinstein cobordism $W_i := \{c_i \leq \phi \leq c_{i+1}\}$ is elementary (meaning that there is no X-trajectory connecting two critical points of ϕ) and the attaching spheres of all critical handles in W_i form a loose Legendrian *link* in $\phi^{-1}(c_i)$.

This notion of flexibility is equivalent to Definition 2.131: any loose Legendrian knot is a loose Legendrian link, hence a flexible Weinstein domain (in the sense of Definition 2.131) is also flexible in this new sense. Conversely, up to Weinstein homotopy, one can assume that all critical values of ϕ are distinct [CE12, Lemma 12.20]. Since the components of a loose Legendrian link are also loose, such a Weinstein domain is explicitly flexible.

Remark 2.133. Let us stress that while a loose Legendrian link is the union of loose Legendrian knots, the converse is not true in general: the union of disjoint loose Legendrian knots need not be loose, since their loose charts could intersect.

In particular, every subcritical Weinstein domain is flexible. In dimension $2n = 4$, the converse is also true since there are no loose Legendrians in dimension 3.

Remark 2.134. The property of being subcritical is not preserved under Weinstein homotopy: one can perform, for example, a Weinstein homotopy which creates two additional critical points with indices $n-1$ and n, respectively [CE12, Proposition 12.21].

The definition of a flexible Weinstein filling is analogous to Definition 2.83.

Definition 2.135. *A* flexible Weinstein filling *of a contact manifold* $(M, \xi = \ker \alpha)$ *is a Weinstein filling* (W, ω, X, ϕ) *such that* W *is a flexible Weinstein domain.*

Finally, we come to the much-announced h-principle of flexible Weinstein structures.

Theorem 2.136 ([CE12, Theorem 14.5])**.** *Two flexible Weinstein structures in dimension* $2n > 4$ *on the same manifold whose symplectic forms are homotopic as non-degenerate* 2*-forms are Weinstein homotopic.* $\square$

For the second theorem, recall that a *diffeotopy* is a smooth homotopy of diffeomorphisms, i.e. a diffeotopy between two diffeomorphisms $f, g\colon M \to M$ on a smooth manifold M is a smooth map $h\colon M \times [0,1] \to M$ such that $h(\cdot, 0) = f$, $h(\cdot, 1) = g$ and each map $h(\cdot, t)\colon M \to M$ is a diffeomorphism.

Theorem 2.137 ([CE12, Theorem 14.7])**.** *Every diffeomorphism* $f\colon W_1 \to W_2$ *between two flexible Weinstein manifolds* $(W_i, \omega_i, X_i, \phi_i)$ *of dimension* $2n > 4$ *for* $i = 1, 2$ *such that* $f^*\omega_2$ *is homotopic to* ω_1 *through non-degenerate* 2*-forms is diffeotopic to an exact symplectomorphism.* $\square$

Let us stress the remarkable nature of these results: a purely topological condition (a homotopy of non-degenerate 2-forms) implies a statement about symplectic structures.

3. Symplectic and positive symplectic homology

In this chapter, we introduce *symplectic homology* which is an invariant of Liouville domains, its variant of *positive symplectic homology* and explain a few of its key properties. Symplectic homology for Liouville domains generalises *Hamiltonian Floer homology*; since some technical details for symplectic homology are motivated by technical issues when generalising the definition of Hamiltonian Floer homology, we begin by presenting the definition of Hamiltonian Floer homology.

3.1. Review of Hamiltonian Floer homology

Hamiltonian Floer homology was introduced in a breakthrough by Andreas Floer to prove a special case of the Arnold conjecture [Flo86; Flo89]; all further progress on the conjecture was essentially obtained by lifting technical restrictions in his methods. The key object in all further considerations are *Hamiltonian vector fields*: to any family of smooth functions on a symplectic manifold, one associates a time-dependent vector field.

Lemma/Definition 3.1. *Let (M, ω) be a symplectic manifold and $H\colon \mathbb{R} \times M \to \mathbb{R}$ be a 1-parameter family of smooth functions such that $H(t, x) = H(t + 1, x)$ for all t, x and denote $H_t := H(t, \cdot)$ for all $t \in \mathbb{R}$. There is unique smooth time-dependent vector field X_{H_t} which satisfies the condition $\omega(X_{H_t}, \cdot) = -dH_t$ for all t.*[1] *The vector field X_{H_t} is called the (time-dependent)* Hamiltonian vector field *for the (time-dependent)* Hamiltonian $H = (H_t)$. □

Since (H_t) is 1-periodic in t, we will often consider $H = (H_t)_{t\in\mathbb{R}}$ as a smooth map $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$. The following definition makes sense whenever the Hamiltonian vector field X_{H_t} is complete. This is automatically true for closed manifolds.

Definition 3.2. *If $(H_t\colon M \to \mathbb{R})$ is a time-dependent Hamiltonian on (M, ω), and X_{H_t} is complete, a* Hamiltonian orbit *of H_t is a smooth curve $\gamma\colon \mathbb{R} \to M$ which satisfies the equation*

$$\gamma'(t) = X_{H_t}(\gamma(t)).$$

[1] This is one of the places where there are mutually inconsistent sign conventions in the literature. While the author believes that this convention is the correct one (see [Wen15] for justification), not all papers we reference use the same convention. See Appendix A.1 for details.

Since the map $H \mapsto X_H$ is a linear map, we observe that

Observation 3.3. The space of all time-independent Hamiltonian vector fields on a given symplectic manifold is linear and in particular contractible. $\square$

The *Arnold conjecture* is a statement about the number of 1-periodic orbits of a time-dependent Hamiltonian. This problem can be attacked using methods similar to Morse theory. We briefly encountered the classical approach to Morse theory in Section 2.3; see Milnor's book [Mil63] for a taste. This approach cannot work for this problem, since we will encounter infinite-dimensional spaces. However, a modern approach to Morse theory generalises formally: one can use a Morse function on a smooth manifold to define homology groups, using the critical points of the Morse function to define the chain groups and define the differential using the flow of a suitable gradient-like vector field (more precisely, a *Morse-Smale system* [AD14, p. 38]). The stable manifold of a gradient-like vector field (Definition 2.123) belongs to this modern approach.

Conley and Zehnder [CZ83] proved the Arnold conjecture for the standard torus $\mathbb{T}^{2n}$ by using ideas similar to modern Morse theory. They showed that the 1-periodic Hamiltonian orbits are the critical points of a suitable function on an infinite-dimensional space. For the torus, this problem can be reduced to finding critical points of a function on a finite-dimensional space, which Conley and Zehnder could attack by using the usual gradient flow. While this strategy works for some examples, their approach of reducing to finite dimensions is not feasible for general symplectic manifolds.

Floer found a proof strategy which was applicable to much wider classes of symplectic manifolds. Floer approach does not require a dimensional reduction like Conley and Zehnder did. This goes in line with a more versatile interpretation of the term "gradient flow", which considers solutions of a suitable partial differential equation (the *Floer equation*, (F) below) instead of gradient trajectories. Implementing this approach brings significant technical obstacles, whose resolution requires a large amount of analytical machinery, and significant technical assumptions. The extent to which these additional assumptions can be removed is one of the major problems in symplectic topology.

We will only present the big picture, and refer the reader to Jean Gutt's thesis [Gut14, Section 1.1] for a crisp overview and to Audin and Damian's textbook [AD14] for a very detailed account. For a more detailed history of the problem and a better account of Conley and Zehnder's proof, we refer both to Hofer and Zehnder's textbook [HZ11, Chapter 6] and to Zehnder's own account [Zeh19].

Let us outline the construction of Hamiltonian Floer homology. To ease notation, we will fix a few assumptions/conventions for the remainder of this section.

Definition 3.4. *A symplectic manifold* (M, ω) *is called* symplectically aspherical *if and only if for any smooth map* $f\colon \mathbb{S}^2 \to M$*, one has the relation* $\int_{\mathbb{S}^2} f^*\omega = 0$*.*

Convention. Let (M, ω) be a closed symplectic manifold which is symplectically aspherical and whose first Chern class (see Definition 2.42) vanishes.

These assumptions are present for technical reasons and can (to some degree) be lifted, at the expense of additional effort. See Remark 3.24 for details.

The first step for the construction was known long before Floer.[2] 1-periodic Hamiltonian orbits are precisely the critical points of a suitable functional, called *Hamiltonian action functional*. (In the literature, one can find the term symplectic action functional also.) We begin with describing its domain.

Definition 3.5. *Consider the space $C^\infty([0,1], M)$ of smooth maps $[0,1] \to M$, endowed with the C^∞_{loc}-topology. The* contractible loop space *of M is the space*

$$\Omega_0(M) := \{\gamma \in C^\infty([0,1], M) \,|\, \gamma(0) = \gamma(1), \gamma \text{ is contractible in } M\},$$

which inherits a natural topology as a subspace of $C^\infty([0,1], M)$.

The space $C^\infty([0,1], M)$ is not a finite-dimensional manifold, but can be shown to be an infinite-dimensional Fréchet manifold.[3] One can show that $\Omega_0(M)$ is an open subset of the closed subspace $\{\gamma \in C^\infty([0,1], M) \colon \gamma(0) = \gamma(1)\}$; in particular it is also infinite-dimensional.

Definition 3.6. *The* Hamiltonian action functional $\mathcal{A}_H \colon \Omega_0(M) \to \mathbb{R}$ *is defined by*

$$A_H(\gamma) := + \int_{\mathbb{D}^2} u^*\omega - \int_{\mathbb{S}^1} H(t, \gamma(t))\, \mathrm{d}t,$$

where $u \colon \mathbb{D}^2 \to M$ is a smooth extension of γ, i.e. we have $\gamma(t) = u(e^{2\pi i t})$ for all $t \in [0,1]$. The map u is called a spanning disc *for γ.*

A few remarks are in order. Firstly, any element $\gamma \in \Omega_0(M)$ is contractible by definition, hence admits an extension u as in the definition. Secondly, the action functional is well-defined because M is symplectically aspherical: given two spanning discs $u, v \colon \mathbb{D}^2 \to M$, gluing u with the map $\overline{v}$ obtained by giving the disc the opposite orientation yields a smooth map $w \colon \mathbb{S}^2 \to M$ which describes the difference of actions, hence one obtains

$$\int_{\mathbb{D}^2} u^*\omega - \int_{\mathbb{D}^2} v^*\omega = \int_{\mathbb{D}^2} u^*\omega + \int_{\mathbb{D}^2} \overline{v}^*\omega = \int_{\mathbb{S}^2} w^*\omega = 0$$

since M is symplectically aspherical by hypothesis. Finally, we alert the reader that there are different sign conventions in use in the literature, and point to Appendix A.1 for details.

The action functional turns out to be a smooth map on the Fréchet manifold $\Omega_0(M)$. There is a well-defined notion of differential and critical points of $\mathcal{A}_H$; the first surprise is that critical points of the Hamiltonian action functional correspond to 1-periodic Hamiltonian orbits.

[2] In Hamiltonian mechanics, a theory in physics which motivated the creation of symplectic geometry, this is known as the *principle of least action*.

[3] The precise definition of a Fréchet manifold will not be used in the sequel.

Lemma 3.7 (e.g. [AD14, Proposition 6.3.3])**.** *A loop $\gamma \in \Omega_0(M)$ is a critical point of $\mathcal{A}_H$ if and only if γ is a 1-periodic Hamiltonian orbit.* $\square$

If γ is a 1-periodic orbit of X_{H_t} and ϕ_t denotes the (time-dependent) flow of X_{H_t}, then $x = \gamma(0)$ is a fixed point of the flow after time 1, and the differential $(d\phi_1)_x$ of ϕ_1 at x yields an endomorphism of T_xM which is called the *Poincaré return map.*

Definition 3.8. *A 1-periodic Hamiltonian orbit γ of X_{H_t} is called* non-degenerate *if and only if 1 is not an eigenvalue of the Poincaré return map. A time-dependent Hamiltonian (H_t) is called* non-degenerate *if and only if all of its 1-periodic Hamiltonian orbits are non-degenerate.*

The following result is important; we will encounter a similar result in the context of Proposition 3.67.

Proposition 3.9 ([AD14, p. 516, Exercise 6])**.** *If (H_t) is a non-degenerate Hamiltonian on (M, ω), the 1-periodic orbits of (H_t) are isolated. In particular, since M is compact there are only finitely many of them.* $\square$

Going forward, we will only consider non-degenerate Hamiltonians. Denote the set of contractible 1-periodic orbits of $H = (H_t)$ by $\mathcal{P}(H)$.

Convention. In addition to the convention above, let $(H_t) = H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ be a non-degenerate time-dependent Hamiltonian on (M, ω).

The next step is to define an analogue of the Morse index. This is subtle: while the Hamiltonian action functional has a well-defined Hessian at each critical point, there are infinite-dimensional subspaces on which the Hessian is negative definite (and the same is true for positive definite subspaces)—hence using the same definition as for Morse theory would yield an infinite index (and coindex[4]), which is of no use. Floer, however, realised that one can still assign a useful index to each critical point. This index is *relative*, meaning that the precise value of each index is unimportant (and depends on some auxiliary choices), whereas the index difference of two critical points is a finite number that *has* a precise meaning. This index is called the *Conley-Zehnder index* of the critical point and is defined as follows:

Given a non-degenerate 1-periodic Hamiltonian orbit γ, one extends γ to a spanning disc $u\colon \mathbb{D}^2 \to M$ and chooses a symplectic trivialisation of the pullback bundle u^*TM. Then, for each $t \in \mathbb{R}$, the differential $d\phi_t\colon T_{\gamma(0)}M \to T_{\gamma(t)}M$ is a symplectic linear map, and composing with the trivialisation yields a smooth path of symplectic matrices $A(t) \in \mathrm{Sp}(2n)$. Because γ is non-degenerate, the endpoint $A(1)$ of the path has no eigenvalue 1, i.e. satisfies $\det(A(1) - \mathrm{id}) \neq 0$. To any such path of symplectic matrices,

[4]The *Morse coindex* of a critical point p of a Morse function ϕ is the Morse index of p for $-\phi$, i.e. the maximal dimension of linear subspace on which the Hessian $\mathrm{Hess}_p\,\phi$ at p is positive definite.

one can assign an integer called the *Conley-Zehnder index*; one then defines the Conley-Zehnder index $\mu_{CZ}(\gamma)$ of the orbit γ as the Conley-Zehnder index of the path $A(t)$. See Gutt [Gut14, Section 6.2] or Audin-Damian [AD14, Chapter 7] for the details.[5]

One can show (see e.g. [AD14, Theorem 7.1.1]) that the choice of trivialisation doesn't matter, since all trivialisations are homotopic and the index defined in the second step depends only on the homotopy class of the path $A(t)$. The choice of the extension u, however, *does* matter in general: in effect, this amounts to choosing a trivialisation of the bundle γ^*TM; two such choices differ by what is called a *relative Chern number*. This is treated in e.g. [Wen16, Exercise 5.3]. Since we assumed that $c_1(M) = 0$, this choice is immaterial in our context and $\mu_{CZ}(\gamma) \in \mathbb{Z}$ is a well-defined integer.

Proposition 3.10 ([Flo89, Proposition 2b; Wen16, Exercise 5.3])**.** *Let u and u' be two spanning discs for x, and $f\colon \mathbb{S}^2 \to M$ be the map obtained by glueing u and u' together. Then the Conley-Zehnder indices obtained via u and u', respectively, differ by the number $2\langle c_1(M), f\rangle \in \mathbb{Z}$.* □

The chain groups for defining Floer homology are just the free abelian groups generated by the contractible 1-periodic Hamiltonian orbits $\mathcal{P}(H)$, with a $\mathbb{Z}$-grading by the Conley-Zehnder index; see Definition 3.22 below.

Next, we want to define the corresponding differential. In Morse homology, one defines the differential by counting trajectories connecting critical points given by the gradient-like vector field. To define a gradient flow for the action functional, one needs to choose a metric on the loop space $\Omega_0(M)$. Let $J = (J_t)_{t\in\mathbb{S}^1}$ be a smooth time-dependent family of compatible almost complex structures on (M, ω), so $g_t := \omega(\cdot, J_t\cdot)$ is a smooth family of Riemannian metrics. Using the g_t, one defines an L^2-product on the tangent space at any loop $\gamma \in \Omega_0(M)$. With this inner product, a map $u\colon \mathbb{R} \times \mathbb{S}^1 \to M$ is a trajectory of the formal *positive*[6] gradient flow of $\mathcal{A}_H$ if and only if it satisfies the *Floer equation*,

$$\frac{\partial u}{\partial s}(s,t) + J_t(u(s,t))\Big(\frac{\partial u}{\partial t}u(s,t) - X_{H_t}(u(s,t))\Big) = 0. \tag{F}$$

We call a map u satisfying (F) a *Floer trajectory.* The next important question is whether a Floer trajectory needs to converge to critical points of the action functional as $s \to \pm\infty$. Unlike in Morse theory, the answer is not always yes, but is still manageable.

Definition 3.11. *The* energy *of a Floer trajectory $u\colon \mathbb{R} \times \mathbb{S}^1 \to M$ is defined as*

$$E(u) := \frac{1}{2}\int_{\mathbb{R}\times\mathbb{S}^1} (|\partial_s u|^2 + |\partial_t u - X_H \circ u|^2)\,\mathrm{d}s\,\mathrm{d}t = \int_{\mathbb{R}\times\mathbb{S}^1} |\partial_s u|^2\,\mathrm{d}s\,\mathrm{d}t.$$

[5] In the literature, there is also a sign choice in the definition of the Conley-Zehnder index. We follow Lazarev's conventions, which agrees with Gutt [Gut14], but has the opposite sign of Audin and Damian [AD14]. See Appendex A.1 for details.

[6] This is again a particular sign convention; see Appendix A.1.

Observe that the energy of a Floer trajectory is non-negative; it is zero if and only if $u(s,t) = \gamma(t)$ is independent of s and a 1-periodic Hamiltonian orbit. Whether a Floer trajectory converges to a periodic orbit at each end depends on its energy.

Proposition 3.12 ([Flo89] or [AD14, Theorem 6.5.6]). *For a Floer trajectory u, we have $E(u) < \infty$ if and only if there exist 1-periodic orbits $x, y \in \mathcal{P}(H)$ such that $\lim_{s\to-\infty} u(s,t) = x(t)$ and $\lim_{s\to\infty} u(s,t) = y(t)$ uniformly in t.* $\square$

The energy of a converging Floer trajectory is determined by its asymptotic behaviour.

Proposition 3.13 (e.g. [AD14, Remark 6.2.2]). *If u is a Floer trajectory which converges to $x \in \mathcal{P}(H)$ resp. $y \in \mathcal{P}(H)$ at $\pm\infty$, then $E(u) = \mathcal{A}_H(y) - \mathcal{A}_H(x)$.* $\square$

We note that since we consider positive gradient flow, the action is increasing along a Floer trajectory, and we get opposite signs compared to e.g. [AD14].

Since we are only interested in contractible periodic orbits, we restrict our attention to contractible Floer trajectories: a Floer trajectory is called *contractible* if and only if the map $u(s,\cdot)\colon \mathbb{S}^1 \to M$ is contractible for one, hence all, $s \in \mathbb{R}$. To summarise, we are interested in the space

$$\mathcal{M} := \{u \in C^\infty(\mathbb{R} \times \mathbb{S}^1, M) \colon u \text{ solves (F)}, u \text{ is contractible and } E(u) < \infty\}.$$

The moduli space $\mathcal{M}$ admits a natural topology in which a sequence converges if and only if it converges in the C^∞_{loc}-topology. Using elliptic regularity, one can show that for Floer cylinders asymptotic to isolated periodic orbits, C^∞_{loc}-convergence also implies uniform convergence near infinity. We will not explain the details since they require significant technical effort.

To define the differential for the Floer chain complex, we want to count Floer trajectories connecting two given periodic orbits, i.e. we want to “count” the number of elements in spaces of the form

$$\mathcal{M}(x,y,H,J) := \{u \in \mathcal{M} \colon \lim_{s\to\infty} u(s,t) = x(t) \text{ and } \lim_{s\to\infty} u(s,t) = y(t)\}$$

for 1-periodic Hamiltonian orbits $x, y \in \mathcal{P}(H)$. We certainly know how to count elements of a compact 0-dimensional manifold. More generally, one might hope that the space $\mathcal{M}(x,y,H,J)$ were a finite-dimensional manifold (w.r.t. its natural topology)..

Whether this is true depends on the data (H,J)—there are examples (of a closed manifold (M,ω) with data (H,J)) when this is *not* true! However, the this condition *does* hold if one changes the almost complex structure J slightly.[7] Let us make that precise.

Definition 3.14. *Let X be a topological space. A subset $A \subset X$ is called* comeagre *if and only if A contains the intersection of a countable family of open and dense subsets, i.e. if and only if there exist $A_n \subset X$, $n \in \mathbb{N}$ which are open and dense such that $\bigcap_{n\in\mathbb{N}} A_i \subset A$.*

[7]Equivalently, one can perturb the Hamiltonian H; this is the approach historically taken by Floer.

In contrast to merely dense sets, the finite or countable intersection of comeagre sets is comeagre again. Observe that comeagre sets are “large”, since they are dense.

Theorem 3.15 (Baire Category Theorem, e.g. [Gen, Theorem 1.2])**.** *Let X be a locally compact topological space or a complete metric space. Then any comeagre subset of X is dense.* □

Theorem 3.16 ([Flo89; AD14])**.** *For every non-degenerate time-dependent Hamiltonian $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$, there is a comeagre set $\mathcal{J}^{reg}(H) \subset \mathcal{J}(M, \omega)$ of compatible almost complex structures such that for all contractible 1-periodic Hamiltonian orbits $x, y \in \mathcal{P}(H)$ and all $J \in \mathcal{J}^{reg}(H)$, the space $\mathcal{M}(x, y, H, J)$ is a smooth manifold of dimension $\mu_{CZ}(y) - \mu_{CZ}(x)$.* □

Results of this kind are often referred to as “transversality” results, since their proof uses the implicit function theorem (in the infinite-dimensional version, i.e. for a smooth map between Banach manifolds) and the necessary condition amounts to a smooth section of a Banach space bundle being transverse to the zero section. For space reasons, we will not explain what this means, let alone the proof.

We will call a pair (H, J) of a non-degenerate Hamiltonian H and an almost complex structure $J \in \mathcal{J}^{\mathrm{reg}}(H)$ a *regular pair*, and from now on we consider only regular pairs.

Note that the space $\mathcal{M}$ has a natural free $\mathbb{R}$-action by shifting the s-coordinate, which yields an action on each space $\mathcal{M}(x, y, H, J)$—in particular, $\mathcal{M}(x, y, H, J)$ cannot be compact and we will never obtain a finite set of points. However, for a regular pair (H, J), this action is proper, hence the quotient $\mathcal{M}(x, y, H, J)/\mathbb{R}$ by this action is a manifold of dimension $\mu_{CZ}(y) - \mu_{CZ}(x) - 1$. This quotient is compact for $\mu_{CZ}(y) - \mu_{CZ}(x) = 1$; we will outline the proof now. To show compactness, we study the limiting behaviour of a sequence (u_k) in $\mathcal{M}$. At first sight, the story seems clear.

Theorem 3.17 (e.g. [AD14, Theorem 6.5.4])**.** *The space $\mathcal{M}$ is compact (in the C^∞_{loc}-topology).* □

However, upon a closer look this statement is not very strong, since the C^∞_{loc}-topology only governs the behaviour on any compact set. Phenomena which “stretch out to infinity” cannot be observed under this topology. Indeed, if we allow shifts of the sequence (u_k), we can observe another phenomenon called *breaking*: every sequence of Floer cylinders has a subsequence which converges to a *broken Floer cylinder*.

Definition 3.18. *A* broken Floer cylinder *is a k-tuple $v = (v^{(0)}, \dots, v^{(k)})$ of Floer trajectories $v^{(i)} \in \mathcal{M}(x_{i-1}, x_i, H, J)$ for $i = 1, \dots, k$, where $x_0, \dots, x_k$ are critical points of $\mathcal{A}_H$. Note that a broken Floer cylinder with $k = 1$ is just a Floer trajectory.*

Theorem 3.19 ([Flo89] or e.g. [AD14, Theorem 9.1.6])**.** *Let (u_n) be a sequence in $\mathcal{M}(x, y, H, J)$. There exist a broken Floer cylinder $v = (v^{(0)}, \dots, v^{(k)})$ with $v^{(i)} \in$*

$\mathcal{M}(x_{i-1}, x_i, H, J)$ *and* $x_0 = x, x_k = y$*, a subsequence* (u_{n_l}) *of* (u_n)*, and sequences* $(s_n^i)_{i=1,\ldots,k}$ *such that for all* i*, we have*

$$u_{n_l}(s_{n_l}^i + \cdot, \cdot) \to v^{(i)} \text{ in } C^\infty_{loc}. \qquad \square$$

For critical points of index difference 1, this breaking cannot occur: since the quotient $\mathcal{M}(x, y, H, J)/\mathbb{R}$ is a smooth manifold of dimension $\mu_{CZ}(y) - \mu_{CZ}(x) - 1$, there is a Floer trajectory connecting two critical points x and y if and only if $\mu_{CZ}(y) - \mu_{CZ}(x) \geq 1$. Hence, we obtain the following result.

Corollary 3.20. *If* (H, J) *is a regular pair and* $\mu_{CZ}(y) - \mu_{CZ}(x) = 1$*, the space* $\mathcal{M}(x, y, H, J)/\mathbb{R}$ *is a finite set of points.* $\square$

Each point comes with a sign induced by a system of *coherent orientations* [FH93]. We omit the details; Gutt's thesis [Gut14, Section 1.1.1] contains a high-level overview. Without these orientations, one still has a well-defined theory with $\mathbb{Z}_2$-coefficients.

Hence, we can define a differential ∂ on the chain groups by counting these points with signs; see Definition 3.23 below. To show that this defines a chain complex, we must prove $\partial^2 = 0$. This follows from a converse to Theorem 3.19—showing that every broken Floer cylinder occurs as the limit of a sequence in $\mathcal{M}(x, y, H, J)$. This is called *gluing*. We will skip the details and just note the following.

Proposition 3.21. *Let* (H, J) *be a regular pair and* $x, y \in \mathcal{P}(H)$ *with* $\mu(y) - \mu(x) = 2$*. Then the quotient space* $\mathcal{M}(x, y, H, J)/\mathbb{R}$ *is a* 1*-dimensional smooth manifold, and its boundary is given by*

$$\partial\mathcal{M}(x, y, H, J)/\mathbb{R} = \bigcup_{z \in \mathcal{P}(H),\, \mu_{CZ}(y) - \mu_{CZ}(z) = 1} \mathcal{M}(x, z, H, J)/\mathbb{R} \times \mathcal{M}(z, y, H, J)/\mathbb{R}. \qquad \square$$

Plugging this into the formula for the differential (as given below), the coefficient of $\langle x \rangle$ in $\partial^2(\langle y \rangle)$ is given by

$$\sum_{z \in \mathcal{P}(H),\, \mu_{CZ}(y) - \mu_{CZ}(z) = 1} \#\mathcal{M}(x, z, H, J)/\mathbb{R} \;\; \#\mathcal{M}(z, y, H, J)/\mathbb{R},$$

where $\#$ denotes a count of points with signs as determined by a system of coherent orientations. The theory of coherent orientations implies that the coefficient vanishes [Gut14, Section 1.1.1]. For $\mathbb{Z}_2$-coefficients, the vanishing follows directly from the gluing theorem and the fact that a compact 1-dimensional manifold has an even number of boundary points; the argument for integer coefficients is similar in spirit to the fact that counting the boundary points with signs gives 0 [Mil97, Chapter 5, Lemma 1].

To summarise, the following definition yields a chain complex.

Definition 3.22. *The* Floer chain complex *associated to a regular pair* (H, J) *is the chain complex* $(FC_*(H, J), \partial)$ *defined as follows. The chain groups are free abelian groups generated by the contractible* 1*-periodic orbits of* X_{H_t}*, graded by the Conley-Zehnder index:*

$$FC_k(H, J) = \bigoplus_{\substack{\gamma \in \mathcal{P}(H) \\ \mu(\gamma) = k}} \mathbb{Z}\langle\gamma\rangle.$$

The differential $\partial \colon FC_*(H, J) \to FC_{*-1}(H, J)$ *defined on the generators by*

$$\partial(\langle y \rangle) = \sum_{\substack{x \in \mathcal{P}(H) \\ \dim \mathcal{M}(x,y,H,J) = 1}} \# \mathcal{M}(x, y, H, J)/\mathbb{R} \; \langle x \rangle$$

and extended by linearity; here $\#$ *denotes a count of points with signs as determined by a system of coherent orientations.*

Hamiltonian Floer homology is defined as the homology of that complex.

Definition 3.23. *The* Hamiltonian Floer homology *associated to a regular pair* (H, J) *on a closed symplectic manifold* (M, ω) *is the homology of its Floer chain complex:* $FH_*(H, J) := H_*(FC_*(H, J), \partial)$.

Remark 3.24. We can now comment about the significance of the assumptions made in this section. If the closed symplectic manifold (M, ω) is not symplectically aspherical, the symplectic action functional depends on the chosen spanning disc, hence one needs to define the action functional on the universal cover of the contractible loop space, i.e. for a pair of a loop and a spanning surface. We also need symplectic asphericality (or some weaker assumption) to prevent *bubbling*, see Discussion 3.33 below.

If one relaxes the assumption on the first Chern class, the Conley-Zehnder index is not a well-defined integer any more, but only defined up to some integer $2N$, where N is called the *minimal Chern number*. In particular, one needs to deal with the different possible choices of trivialisations; the nicest way of doing so is to consider a so-called *Novikov ring* which captures these possible choices; the Hamiltonian Floer homology then becomes a module over that ring.

Finally, one can also consider non-contractible loops: for any free homotopy class of loops $\mathbb{S}^1 \to M$, one can define Hamiltonian Floer homology for loops in that homotopy class. In that case, one fixes a reference loop in that class, and considers pairs of a loop and a spanning surface, i.e. of maps $\sigma \colon \Sigma \to M$, where Σ is a compact surface with two oriented boundary components and σ coincides with the reference loop resp. γ on the boundary components. (For contractible loops, the reference loop was the constant loop, hence one could consider $\mathbb{D}^2$ as spanning surface.)

Finally, we investigate whether the Hamiltonian Floer homology $FH(H, J)$ of (M, ω) depends on the choice of H or J. We would like $FH(H, J)$ to be independent of J

(since we chose any regular J, there was no geometric meaning to our choice) and also independent of H (to obtain an invariant of (M, ω)).

Let (H_1, J_1) and (H_2, J_2) be regular pairs. We will construct a map $FH(H_1, J_1) \to FH(H_2, J_2)$. The first step is to choose a "homotopy" (H_s, J_s) from (H_1, J_1) to (H_2, J_2), more precisely a smooth family $(H_s, J_s)_{s\in\mathbb{R}}$ of (still time-dependent!) Hamiltonians H_s and compatible almost complex structures $J_s \in \mathcal{J}(M, \omega)$ such that for some $C > 0$, we have

$$H_s \equiv H_1 \text{ and } J_s \equiv J_1 \text{ for } s < -C, \quad H_s \equiv H_2 \text{ and } J_s \equiv J_2 \text{ for } s > C. \tag{3.1}$$

We will sometimes write these conditions as $H_s \equiv H_1$ for $s \ll 0$; similarly for the others.

Such a homotopy always exists: since $\mathbb{R}$ is contractible, there is a smooth path of Hamiltonians H_s from H_1 to H_2. By Proposition 2.11, there is also a smooth path of compatible almost complex structures from J_1 to J_2.

The first key insight is to consider solutions $u\colon \mathbb{R} \times \mathbb{S}^1 \to M$ to the *s-dependent Floer equation*

$$\partial_s u + (J_{s,t} \circ u)\left(\partial_t u - X_{H_{s,t}}(u)\right) = 0. \tag{Fs}$$

Energy is defined just as for s-independent Floer trajectories (see Definition 3.11). It turns out that a finite energy solution to the s-dependent Floer equation converges to 1-periodic Hamiltonian orbits at either end: to an orbit of H_1 at the negative and to an orbit of H_2 at the positive end. Hence, we can use these solutions to define a chain map.

Proposition 3.25 (e.g. [AD14, Theorem 11.1.1][8])**.** *Suppose (H_1, J_1) and (H_2, J_2) are regular pairs and $\{H_s, J_s\}$ is a smooth family of Hamiltonians resp. compatible almost complex structures which satisfy the conditions (3.1). If u is a solution of (Fs) and $E(u) < \infty$, there exist $x \in \mathcal{P}(H_1)$ and $y \in \mathcal{P}(H_2)$ such that $\lim_{s\to-\infty} u(s,t) = x(t)$ and $\lim_{s\to\infty} u(s,t) = y(t)$ uniformly in t.* □

The converse also holds; we will use the following energy bound.

Proposition 3.26 (e.g. [AD14, Proposition 11.1.2])**.** *Let u be a smooth solution to (Fs), let $C > 0$ be chosen so the conditions (3.1) hold and suppose $\lim_{s\to-\infty} u(s,t) = x(t)$ and $\lim_{s\to\infty} u(s,t) = y(t)$ uniformly in t for some $x \in \mathcal{P}(H_1)$ and $y \in \mathcal{P}(H_2)$. Then the energy of u is given as*

$$E(u) = \mathcal{A}_H(y) - \mathcal{A}_H(x) - \int_{[-C,C]\times\mathbb{S}^1} \partial_s H(s,t,u(s,t))\,\mathrm{d}s\,\mathrm{d}t. \tag{3.2}$$

[8]There is a minor difference in that Audin and Damian vary the Hamiltonian instead of the almost complex structure. Still, the details for the transversality, compactness and gluing theorems are essentially the same.

Hence, for a smooth family $\{H_s, J_s\}$ which satisfies the conditions (3.1), for $x \in P(H_1)$ and $y \in \mathcal{P}(H_2)$, we consider the moduli space

$$\mathcal{M}(x, y, \{H_s\}, \{J_s\}) := \Big\{ u \in C^\infty(\mathbb{R} \times \mathbb{S}^1, M) \colon u \text{ solves (Fs)}, \ u \text{ is contractible}, \\ \lim_{s \to -\infty} u(s,t) = x(t) \text{ and } \lim_{s \to \infty} u(s,t) = y(t) \Big\}.$$

Inspired by Proposition 3.25, we would like to define a chain map $\phi \colon FC(H_1, J_1) \to FC(H_2, J_2)$ by the formula

$$\langle x \rangle \mapsto \sum_{\substack{y \in \mathcal{P}(H_2);\ \mathcal{M}(x,y,\{H_s\},\{J_s\}) \\ \text{is a 0-dimensional manifold}}} \#\mathcal{M}(x, y, \{H_s\}, \{J_s\}) \, \langle y \rangle,$$

and go through the same steps as before to make this well-defined. The map ϕ would induce a map on the graded homology groups $\phi_* \colon FH(H_1, J_1) \to FH(H_2, J_2)$. If everything goes well, these maps will be isomorphisms, which would show independence of the data (H, J).

Let us put this plan to action. For time-dependent compatible almost complex structures $J_1, J_2 \in \mathcal{J}(M, \omega)$, consider the space

$$\mathcal{J}^C(M, \omega, J_1, J_2) := \Big\{ J \colon \mathbb{R} \times [0,1] \to \mathcal{J}(M, \omega) \ \Big| \ J(s,t) \equiv J_1(t) \text{ for } s \leq -C, \\ J(s,t) \equiv J_2(t) \text{ for } s \geq C \Big\},$$

for some $C > 0$, endowed again with the C^∞_{loc}-topology. Note the choice of a uniform constant C: in order to apply the Baire Category theorem (Theorem 3.15) later, it is necessary that the space $\mathcal{J}^C(M, \omega, J_1, J_2)$ be a complete metrizable space. Fixing the sizes of both cylindrical ends on which J matches J_1 or J_2 by choosing C uniformly makes sure this holds.

Theorem 3.27 (e.g. [AD14, Theorem 11.1.7]). *Let (H_1, J_1) and (H_2, J_2) be regular pairs, $C > 0$ be a constant and $\{H_s\}$ be a smooth path of Hamiltonians with $H_s \equiv H_1$ for $s \leq -C$ and $H_s \equiv H_2$ for $s \geq C$. There exists a comeagre set $\mathcal{J}^{reg} \subset \mathcal{J}^C(M, \omega, J_1, J_2)$ such that for $J \in \mathcal{J}^{reg}$, the moduli space $\mathcal{M}(x, y, \{H_s\}, \{J_s\})$ is a smooth manifold of dimension $\mu_{CZ}(y) - \mu_{CZ}(x)$.* □

Such homotopies $\{H_s, J_s\}$ with $\{J_s\} \in \mathcal{J}^{\text{reg}}$ are called *regular* homotopies. The next step is a compactness theorem for regular homotopies.

Theorem 3.28 (e.g. [AD14, Corollary 11.1.13]). *If $(\{H_s\}, \{J_s\})$ is a regular homotopy and $\mu_{CZ}(y) - \mu_{CZ}(x) = 0$, the space $\mathcal{M}(x, y, \{H_s\}, \{J_s\})$ is compact.* □

Hence, for a regular homotopy $\{H_s, J_s\}$, the map

$$\phi\colon FC(H_1, J_1) \to FC(H_2, J_2), \langle x\rangle \mapsto \sum_{\substack{y\in P(H_2)\\ \mu_{CZ}(x)=\mu_{CZ}(y)}} \#\mathcal{M}(x, y, \{H_s\}, \{J_s\})\, \langle y\rangle$$

is well-defined. Again, each point in the moduli space $\mathcal{M}(x, y, \{H_s\}, \{J_s\})$ is counted with a sign determined by a system of coherent orientations.

One can show that ϕ is a chain map; this requires an analogue of the gluing theorem. Consequently, we obtain a well-defined morphism $\phi_*\colon FH(H_1, J_1) \to FH(H_2, J_2)$.

Remark 3.29. While the formulas for the Floer differential and these chains maps look very similar, there is an important difference: the Floer equation (F) and hence the moduli spaces $\mathcal{M}(x, y, H, J)$ are invariant under shifts of the s-coordinate, hence we had to quotient by this $\mathbb{R}$-action to obtain a 0-dimensional manifold. As a consequence, the Floer differential has degree -1. In contrast, the s-dependent Floer equation (Fs) and the spaces $\mathcal{M}(x, y, \{H_s\}, \{J_s\})$ are *not* invariant under shifts of the s-coordinate, hence there is no $\mathbb{R}$-action and the chain maps yield a degree-preserving continuation map.

A priori, the chain map ϕ depends on the homotopy (H_s, J_s). However, one can show that interpolating between two choices of regular homotopies (a “homotopy of homotopies”) yields a chain homotopy [AD14, Proposition 11.2.8], in a similar way as for the continuation maps. Thus, different homotopies induce the same map ϕ_*. Next, one shows that the continuation maps respect composition.

Theorem 3.30 (e.g. [AD14, Proposition 11.2.9])**.** *Let (H_1, J_1), (H_2, J_2) and (H_3, J_3) be regular pairs. Making any choice of regular homotopies between (H_1, J_1), (H_2, J_2) and (H_3, J_3), the corresponding maps in homology $\phi^{12}_*\colon FH(H_1, J_1) \to FH(H_2, J_2)$, $\phi^{23}_*\colon FH(H_2, J_2) \to FH(H_3, J_3)$ and $\phi^{13}_*\colon FH(H_1, J_1) \to FH(H_3, J_3)$ satisfy the relation*

$$\phi^{23}_* \circ \phi^{12}_* = \phi^{13}_*. \qquad \square$$

Finally, one shows that the constant homotopy yields the identity map [AD14, Proposition 11.1.14].

Remark. To those readers who are well-versed in category theory, we remark that this construction can be seen as a functor between suitable categories. See Section 3.2 for details.

Hence, all the continuation maps are isomorphisms, and we obtain our desired independence of the regular pair (H, J).

Theorem 3.31 ([Flo89])**.** *Let (M, ω) be a $2n$-dimensional closed symplectic manifold which is symplectically aspherical and has $c_1(M) = 0$. The Hamiltonian Floer homology $FH(M, \omega) := FH(H, J)$ is independent of the regular pair (H, J).* $\square$

One can even compute the Hamiltonian Floer homology: up to a shift in grading, it is isomorphic to Morse homology and singular homology.

Theorem 3.32 ([Flo89]). *If (M, ω) is a $2n$-dimensional closed symplectic manifold which is symplectically aspherical and satisfies $c_1(M) = 0$, we have $FH_*(M, \omega) = H_{n-*}(M, \omega)$.* □

For transferring these arguments to Liouville domains, let us recapitulate the key steps for proving this result. If we assume the transversality theorems to follow, the key results were compactness and gluing results: to define Hamiltonian Floer homology, we needed a compactness result for Floer trajectories between critical points of index difference 1, and a gluing result for trajectories with index difference 2. Showing independence of the regular pair used requires similar results for s-dependent Floer trajectories.

Discussion 3.33. All compactness results were based on the same three ingredients.

(1) Solutions $u \in \mathcal{M}(x, y, H, J)$ must satisfy an a priori C^0-bound.

(2) Solutions $u \in \mathcal{M}(x, y, H, J)$ satisfy a uniform bound on their energy $E(u)$.

(3) All possible holomorphic spheres that could bubble off live in spaces of dimension at most $\dim \mathcal{M}(x, y, H, J) - 2$.

Ingredient (1) makes sense—otherwise, curves might "run away" in the target manifold. We already encountered ingredient (2) implicitly: the energy $E(u)$ is clearly continuous in u, hence a sequence (u_k) must have bounded energy $E(u_k)$ if it is to converge to a (possibly broken) Floer cylinder.▷ *I don't find this second argument too convincing any more...* ◁ Explaining ingredient (3) goes a bit further. When discussing the compactification of a moduli space $\mathcal{M}(x, y, H, J)$, another phenomenon called *bubbling* can occur. For a sequence (u_k) of Floer cylinders, there may be a point $\zeta \in \mathbb{R} \times \mathbb{S}^1$ such that $|du_k(\zeta)|$ diverges. By rescaling the u_k near ζ in a clever way, one can show that the u_k near ζ converge to a map on a surface which has an additional component $\mathbb{S}^2$; one says that a sphere has *bubbled off*. If one assumes that M is symplectically aspherical, this already gives a contradiction. For defining the Floer complex, the weaker statement above is sufficient since it shows that bubbling does not influence the Floer complex.

In our case, ingredient (2) follows from energy bounds, as equation (3.2) guarantees a uniform bound on the energy: since M is compact, the Hamiltonians H_s are uniformly bounded. Ingredient (1) is automatic since M is compact. When considering Liouville domains, ensuring these conditions will require a much more careful analysis. However, item (3) will follow directly from the exactness of the symplectic form.

3.2. Direct limits

In the next section we will encounter the concept of a *direct limit*, hence we present its definition here. We refer to Eilenberg and Steenrod's textbook [ES52, Section 8.4] for details. Almost all concepts and results can be defined in greater generality, using the language of category theory. We have phrased the definitions without recourse to this framework as much as possible, but indicate the general phrasing when helpful.

Direct limits involve a collection of objects with an ordering and maps between these objects which are compatible with the ordering. The correct notion of ordering is called a *directed set*.

Recall. A *preorder* on a set A is a transitive and reflexive relation. A *partial order* on A is a preorder which is in addition anti-symmetric.

Example 3.34. On any set A, the trivial relation $\leq$ defined by $a \leq b$ for all $a, b \in A$ is a preorder (since it is trivially transitive and reflexive), but not a partial order (since it is not anti-symmetric).

Definition 3.35. *A* directed set *is a pair $(I, \leq)$ of a set I and a preorder $\leq$ on I with the property that any pair of elements of I has a common upper bound: for all $a, b \in I$, there exists an element $c \in I$ such that $a \leq c$ and $b \leq c$.*

Example 3.36. The standard example is $(\mathbb{N}, \leq)$, the set of natural numbers ordered by height. More generally, any totally ordered set is directed since for every pair of elements, the larger element of the two is an upper bound.

Next, we define the objects we want to take direct limits of.

Definition 3.37. *Let $(I, \leq)$ be a directed set. A* directed system *of abelian groups over $(I, \leq)$ is a collection $(A_i)_{i \in I}$ of abelian groups indexed by the set I together with a collection $(\phi_{ij})_{i,j \in I}$ of group homomorphisms $\phi_{ij} \colon A_i \to A_j$ for all $i, j \in I$ with $i \leq j$ which satisfy the properties*

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij} \text{ and } \phi_{ii} = \mathrm{id}_{A_i} \qquad \text{for all } i, j, k \in I \text{ with } i \leq j \leq k.$$

More generally, for any category $\mathcal{A}$, a directed system *in the category $\mathcal{A}$ over $(I, \leq)$ is a pair $((A_i), (f_{ij}))$, where $(A_i)_{i \in I}$ is a collection of objects indexed by I and (f_{ij}) is a collection of morphism $f_{ij} \colon A_i \to A_j$ for all $i \leq j$ with the properties that*

$$f_{ik} = f_{jk} \circ f_{ij} \text{ and } f_{ii} = \mathrm{id}_{A_i} \qquad \text{for all } i, j, k \in I \text{ with } i \leq j \leq k.$$

Remark. To those readers who like category theory, we remark that a directed system in the category $\mathcal{A}$ over a directed set $(I, \leq)$ is precisely a covariant functor $I \to \mathcal{A}$, where the directed set $(I, \leq)$ is regarded as a category I by taking $\mathrm{Ob}(I) = I$ and $\mathrm{Mor}(i, j) = \begin{cases} \{i_{ij}\} & \text{if } i \leq j \\ \emptyset & \text{otherwise} \end{cases}$ and imposing the obvious composition rules.

Example 3.38. We already encountered a directed system in Section 3.1: the directed set is the set of all regular pairs (H, J), with the trivial preorder from Example 3.34. The objects (A_i) are the graded abelian groups $FH(H, J)$, the morphisms are the continuation maps $FH(H_1, J_1) \to FH(H_2, J_2)$, which indeed behaved functorially.

Now we come to the definition of direct limits. The general categorical definition is a bit involved, but there is a much more elementary construction for modules over a commutative ring R. We present this construction since it is sufficient for our purposes. In the following, let R be a commutative ring with unit.

Definition 3.39 ([ES52, Section 8.4])**.** *Let (A_i, f_{ij}) be a directed system of R-modules over a directed set $(I, \leq)$. Let $Q \subset \bigoplus_{i \in I} A_i$ be the submodule generated by all elements $f_{ij}(x_i) - x_i$ for $i, j \in I$ with $i \leq j$ and $x_i \in A_i$. The direct limit of (A_i, f_{ij}) is defined as the quotient module*

$$\varinjlim A_i := \bigoplus_{i \in I} A_i / Q.$$

Note that for each $i \in I$, the inclusion $A_i \subset \bigoplus_{i \in I} A_i$ induces an R-module homomorphism $\iota_i \colon A_i \to \varinjlim_{i \in I} A_i, x \mapsto [x]$. These maps $(f_i)_{i \in I}$ satisfy $f_i = f_j \circ f_{ij}$ for all $i \leq j$.

Remark. Hence, in Section 3.1 we could have taken the direct limit of the complexes $FH(H, J)$—the result just wouldn't have been interesting. The situation will be different in Section 3.3.

For computing a direct limit, it can be useful to restrict to a smaller directed set.

Definition 3.40. *A subset $A \subset I$ of a directed set $(I, \leq)$ is called a* directed subset *if and only if restricting the preorder on I to A makes A a directed subset. A subset $A \subset I$ is called* cofinal *if and only if any element $x \in I$ has an upper bound in A, i.e. for all $x \in I$, there is an element $a \in A$ with $x \leq a$.*

Observation 3.41. A cofinal subset $A \subset I$ of a directed set $(I, \leq)$ is a directed subset.

Lemma 3.42 ([ES52])**.** *If $(I, \leq)$ is a directed set, $A \subset I$ a cofinal subset and (A_i, f_{ij}) is a directed system over $(I, \leq)$, the objects and maps which correspond to elements in A form a directed system over A.* □

The crucial property is that direct limits can be computed just via any cofinal subset.

Proposition 3.43 ([ES52, Chapter 8, Theorem 4.13])**.** *Let $(A_i, f_{ij})_{i \in I}$ be a directed system of R-modules over a directed set $(I, \leq)$ and $A \subset I$ a cofinal subset. Denote the restricted system by (B_i, g_{ij}), i.e. $B_i = A_i$ and $g_{ij} = f_{ij}$ for all $i, j \in A$. Then the direct limits coincide: there is an R-module isomorphism $\varinjlim B_i \to \varinjlim A_i$.* □

In Section 3.3 we will use a suitable cofinal subset to compute a direct limit.

As it turns out, a map between two directed systems induces a map between the direct limits. In categorical language, forming the direct limit is a *functor*. We refer the reader to [ES52, Chapter 4, Section 2] for the definition of a category and functor.

Definition 3.44. *Let $(M, \leq)$ and $(N, \leq)$ be directed sets. A function $f \colon M \to N$ is called* order-preserving *if and only if $\alpha \leq \beta$ in M implies $f(\alpha) \leq f(\beta)$ in N.*

Definition 3.45. *Given two directed systems* (A_i, f_{ij}) *and* (B_i, g_{ij}) *of* R*-modules over directed sets* $(I, \leq)$ *and* $(J, \leq)$*, a morphism* $\phi\colon (A_i, f_{ij}) \to (B_k, g_{kl})$ *consists of an order-preserving map* $\phi\colon I \to J$ *and for each* $i \in I$*, an* R*-module homomorphism* $\phi_i\colon A_i \to B_{\phi(i)}$ *such that for all* $i \leq j$*, we have* $g_{ij} \circ \phi_i = \phi_j \circ f_{ij}$*. In other words, the diagram on the right commutes.*

$$\begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[d, "\phi_i"] & A_j \arrow[d, "\phi_j"] \\ B_{\phi(i)} \arrow[r, "g_{\phi(i)\phi(j)}"] & B_{\phi(j)} \end{tikzcd}$$

One can easily check that directed systems (of R-modules, and more generally over any category) form a category $\mathcal{D}$. We have the following result.

Theorem 3.46 ([ES52, Chapter 8, Theorem 4.12])**.** *Taking the direct limit is a covariant functor* $\mathcal{D} \to \mathrm{Mod}(R)$*, from the category of directed systems of* R*-modules to the category of* R*-modules.* □

Moreover, the direct limit is an *exact* functor, meaning that it preserves exact sequences.

Theorem 3.47 ([ES52, Chapter 8, Theorem 5.4])**.** *Let* $A \to B \to C$ *be an exact sequence of directed systems of* R*-modules. The corresponding sequence of direct limits is also exact.* □

In Chapter 6, we also need that the direct limit commutes with tensor products. This is a standard exercise in commutative algebra.

Lemma 3.48 ([AM69, Exercise 2.20])**.** *Let* R *be a commutative unital ring,* N *an* R*-module and* $(M_i)_{i \in I}$ *a directed system of* R*-modules. There is a natural isomorphism* $\varinjlim(M_i \otimes_R N) \cong \varinjlim(M_i) \otimes N$*.* □

Remark 3.49. Lest the reader think that all these constructions generalise right away to general categories, a word of caution. Firstly, while the concept of direct limit can be defined in any category, not every directed system needs to have a direct limit. (When the direct limit exists, however, it is unique in a precise sense and again defines a functor.) Secondly, exactness of the direct limit functor for R-modules is a particular property. For example, a dual construction called *inverse limit* is *not* exact in general.

3.3. Symplectic homology

Having reviewed Hamiltonian Floer homology, we want to transfer the same ideas to non-closed symplectic manifolds. This can be done in various settings, including compact symplectic manifolds whose boundary is "of contact type". We will restrict to Liouville domains (which are a special case), since the general case demands attention to additional technical details which we prefer to not worry about. Since Liouville domains are compact, they have some invariants (such as their volume) which are not interesting for our purpose. Hence, we define symplectic homology in terms of the completion (recall Definition 2.61) of a Liouville domain.

Convention. Let $(W, \omega = d\lambda, X)$ be a Liouville domain with $c_1(W) = 0$. Denote $M := \partial W$ and let $\alpha := \lambda|_{\partial W}$ be the corresponding contact form on M. Let $(\widehat{W}, \widehat{\omega})$ be the completion of (W, ω) with respect to the contact form α.

This different setting brings about a few changes when trying to define an analogue of Hamiltonian Floer homology. One such change concerns the ingredients for the necessary compactness results: while the exactness of the symplectic form elegantly resolves any concerns about bubbling, ensuring the necessary a priori bounds for the energy and norm of Floer trajectories becomes an issue and requires a new idea. In the wake of this, the use of a direct limit will become necessary.

However, there is also a beneficial difference compared to closed manifolds: finite type Liouville manifolds always contain a cylindrical end of a symplectisation, and one can even tweak the associated contact form. This provides extra structure to the Hamiltonian orbits and allows us to define an additional invariant called *positive symplectic homology*.

Let us begin by analysing the changes through the presence of cylindrical ends. It turns out that under suitable conditions on the Hamiltonian, the Hamiltonian orbits in a cylindrical end correspond to Reeb orbits on the contact boundary ∂W. The starting point to this result is the following observation.

Lemma 3.50. *Let $(M, \xi = \ker\alpha)$ be a contact manifold, $(\mathbb{R} \times M, d(e^r\alpha))$ its symplectisation and $H\colon \mathbb{R} \times M \to \mathbb{R}$ a Hamiltonian of the form $H(r,p) = h(e^r)$ for some smooth function $h\colon \mathbb{R} \to \mathbb{R}$. Then the Reeb vector field R_α on each level $\{r\} \times M$ and the Hamiltonian vector field X_H are related by*

$$X_H(r,p) = h'(e^r)R_\alpha(p).$$

Proof. Observing that we have $\omega = e^r(dt \wedge \alpha + d\alpha)$, we directly compute

$$\begin{aligned}
\omega(h'(e^r)R_\alpha, \cdot) &= e^r(dt \wedge \alpha)(h'(e^r)R_\alpha, \cdot) + d\alpha(h'(e^r)R_\alpha, \cdot) \\
&= e^r h'(e^r)\Big[dt(R_\alpha)\alpha(\cdot) - \alpha(R_\alpha)dt(\cdot) + d\alpha(R_\alpha, \cdot)\Big] \\
&= -e^r h'(e^r)dt(\cdot)
\end{aligned}$$

using the properties $\alpha(R_\alpha) = 1$ and $d\alpha(R_\alpha, \cdot) = 0$ of a Reeb vector field. Since H has the form $H = h(e^r)$, we have $-dH(\cdot) = -e^r h'(e^r)dt$, hence the claim follows right from Definition 3.1. □

Hence, if we restrict to Hamiltonians of the form $H(r,p) = h(e^r)$, the Reeb and Hamiltonian vector fields are parallel. We would like to deduce a statement about their orbits. This is possible in much greater generality, for general integral curves of vector fields on compact manifolds.[9]

[9] We will see that even though $\widehat{W}$ is not compact, the following results can still be applied in our setting.

Proposition 3.51. *Let M be a compact manifold, let X and Y be vector fields on M such that $Y = fX$ for some smooth function $f \in C^\infty(M)$. Then any integral curve of X has a reparametrisation, unique up to translation, which is an integral curve of Y.*

Proof. Since M is compact, both X and Y are complete vector fields. Let $\gamma\colon \mathbb{R} \to M$ be an integral curve for X. We wish to find a smooth function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $\tilde{\gamma} := \gamma \circ \phi\colon \mathbb{R} \to M$ is an integral curve of Y. By translation, we may assume $\phi(0) = 0$.

Claim. *$\tilde{\gamma}$ is an integral curve of Y starting at $\gamma(0)$ if ϕ solves the ordinary differential equation*

$$\phi(0) = 0, \quad \phi'(t) = (f \circ \gamma)(\phi(t)). \tag{3.3}$$

If $X \neq 0$, the converse also holds.

Proof of Claim. Indeed, for any $t \in \mathbb{R}$, we compute that

$$\tilde{\gamma}'(t) = \gamma'(\phi(t))\, \phi'(t) = X_{\gamma(\phi(t))}\, \phi'(t) = \phi'(t)\, X_{\tilde{\gamma}(t)}.$$

By definition, $\tilde{\gamma}$ is an integral curve of Y if and only if $\tilde{\gamma}'(t) = Y_{\tilde{\gamma}(t))} = f(\tilde{\gamma}(t)\, X_{\tilde{\gamma}(t)}$. If $X \neq 0$, $\tilde{\gamma}$ being an integral curve also implies that $\phi'(t) = f(\tilde{\gamma}(t))$. △

By the existence and uniqueness theorem for ordinary differential equations (see e.g. [Lee02, Theorem 17.9]), such a reparametrisation is unique wherever it exists, and always exists locally. In this case, a solution must even exist globally; one way of proving this is the following. Equation (3.3) defines a smooth vector field on $\mathbb{R}$. Since M is compact, the function $f \circ \gamma$ is bounded, hence ϕ is a bounded vector field on $\mathbb{R}$. Since $\mathbb{R}$ is complete, the result follows from Lemma 3.52. □

Lemma 3.52. *Let M be a complete Riemannian manifold and X be a smooth vector field on M. Suppose there exists a constant $C > 0$ such that $|X(p)| < C$ for all $p \in M$. Then X is complete.* □

The idea of proof is similar to showing that any vector field on a compact manifold is complete: for any integral curve γ of the vector field, one uses completeness of M to exhibit a limit point, and concludes that γ can always be extended. We skip the details.

Combining Lemma 3.50 with Proposition 3.51, we obtain a result for Reeb and Hamiltonian orbits in a symplectisation.

Proposition 3.53. *If γ is a Reeb orbit of M of period $T > 0$, and $T \in \mathbb{R}$ satisfies $h'(e^r) = T$, then $x(t) := (r, \gamma(T\,t))$ is a 1-periodic Hamiltonian orbit in $\mathbb{R} \times M$. Conversely, all non-constant 1-periodic Hamiltonian orbits of $\mathbb{R} \times M$ are of this form.*

Proof. Each Reeb orbit (by definition) is contained in a level set $\{r\} \times M$. The same holds for a non-constant Hamiltonian orbit γ: if $h'(e^r) \neq 0$, this is automatic as Hamiltonian orbits are contained in a level set of the Hamiltonian. If we have $h'(e^r) = 0$ for some point $\gamma(t_0) = (r, p)$, then γ must be constant.

In particular, each Hamiltonian or Reeb orbit is contained in some compact subset $[-T, T] \times M$ and we can apply Proposition 3.51 in combination with Lemma 3.50. □

Stationary Hamiltonian orbits occur precisely at the points (r, p) for which $h'(e^r) = 0$ holds. In particular, such orbits cannot be isolated; they will be excluded when defining symplectic homology.

Let us emphasize that while correct, the discussion above was somewhat misleading: if $H_t \equiv H$ is time-independent, all its orbits come in families parametrised by $\mathbb{S}^1$ and will not be non-degenerate. However, one can show that a suitable small time-dependent perturbation of H turns each $\mathbb{S}^1$-family of orbits of index k into two non-degenerate orbits of almost the same period, with indices k and $k-1$. We will not explain this in detail since the details are rather involved and technical. With this caveat understood, we will mostly suppress the time-dependence from the notation.

Let us now study how to obtain compactness theorems in our setting. In Discussion 3.33, we saw that three ingredients were required to set up a Floer theory. In our setting, there can be no bubbling since the symplectic form is exact. A slightly more conceptual result is the following.

Proposition 3.54. *For an exact symplectic manifold $(W, \omega = d\lambda)$ and a closed surface Σ, any smooth map $u\colon \Sigma \to W$ satisfies $\int_\Sigma u^*\omega = 0$.*

Proof. By Stokes' theorem, we have $\int_\Sigma u^*\omega = \int_\Sigma d(u^*\lambda) = \int_{\partial\Sigma} u^*\lambda = 0$. □

Corollary 3.55. *Every Liouville manifold is symplectically aspherical.* □

The first two ingredients from Discussion 3.33, having a priori bounds for the energy and the C^0 norm of Floer trajectories $u \in \mathcal{M}(x, y, H, J)$, as not as automatic as in Section 3.1. However, one can still deduce these if one imposes some slight assumptions on the Hamiltonian and almost complex structure.

The key statement is a general result known as the *maximum principle*. The statement below is not the most general case, but is sufficient for our purposes. We refer to Gilbarg and Trudinger's book [GT01, Chapter 3.1] for further background.

Definition 3.56. *Let $\Omega \subset \mathbb{R}^n$ be open and connected, $n \geq 2$. Consider a differential operator L of the form*

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) D_{ij}u + \sum_{i=1}^{n} b^i(x) D_i u + c(x)u, \quad \text{with } a^{ij} = a^{ji}$$

for $x = (x_1, \dots, x_n) \in \Omega$ and $u \in C^2(\Omega)$. The operator L is called elliptic *if and only if for each $x \in \Omega$, the coefficient matrix $(a^{ij}(x))$ satisfies the inequality*

$$0 < \lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2 \tag{$*$}$$

for all non-zero vectors $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$, where $\lambda(x)$ and $\Lambda(x)$ are the minimum and maximum eigenvalue of the matrix $a^{ij}(x)$, respectively. In particular, the matrix $(a^{ij}(x))$ must be positive definite for all x.

Example 3.57. For all $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open and connected, the *Laplace operator* $L = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$ is an elliptic operator on Ω. The coefficient matrix $(a^{ij}(x))$ is the identity matrix, hence all terms in $(*)$ are equal to $|\xi|^2$.

Definition 3.58. *Let $U \subset \mathbb{R}^n$ be open. A smooth function $f: U \to \mathbb{R}$ is called* subharmonic *if and only if its Laplacian is non-negative.*

Proposition 3.59 ([GT01, Theorem 3.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set and L an elliptic operator on Ω with $c = 0$. Suppose that $Lu \geq 0$ for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then the maximum of u in $\overline{\Omega}$ is achieved on $\partial\Omega$, i.e. $\sup_{\Omega} u = \sup_{\partial\Omega}$. In particular, u has no local maximum.* □

Corollary 3.60 ([GT01, Theorem 2.2])**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded, open and connected and $u: \Omega \to \mathbb{R}$ be subharmonic. Then u has no local maximum.* □

It turns out that if we impose some mild restrictions on the Hamiltonian H and the almost complex structure J, we can apply the maximum principle to our setting. The first restriction will be on the almost complex structures.

Definition 3.61. *Let $(M, \xi = \ker \alpha)$ be a contact manifold and $(\mathbb{R} \times M, d(e^r\alpha))$ be its symplectisation. A compatible almost complex structure $J \in \mathcal{J}(\mathbb{R} \times M)$ is called* cylindrical *if and only if it satisfies the following properties.*

- *J is invariant under $\mathbb{R}$-translation, i.e. $J_{(t,p)} = J_{(t',p)}$ for all $p \in M$ and $t, t' \in \mathbb{R}$.*
- *$J(\xi) = \xi$ and $J|_\xi$ restricts to a compatible almost complex structure on the symplectic vector bundle $(\xi, d\alpha) \to M$.*
- *$J(\partial_r) = R_\alpha$ and $J(R_\alpha) = -\partial_r$, where ∂_r denotes the unit vector in $\mathbb{R}$-direction.*

Denote $\mathcal{J}(M, \alpha) := \{J \in \mathcal{J}(\mathbb{R} \times M,\, d(e^r\alpha)) \mid J \text{ is cylindrical}\}$.

Definition 3.62. *A time-dependent almost complex structure $J = (J_t)_{t \in \mathbb{S}^1}$ on $\widehat{W}$ is called* admissible *if and only if*

- *$J_t \in \mathcal{J}(\widehat{W}, \widehat{\omega})$ for all t, i.e. each J_t is everywhere compatible with $\widehat{\omega}$*
- *Every J_t matches a common time-independent almost complex structure $J \in \mathcal{J}(M, \alpha)$ on $[T, \infty) \times M$ for some $T \geq T_0$.*

We denote the space of all admissible almost complex structures on $(\widehat{W}, \widehat{\omega})$ by $\mathcal{J}(W, \omega, \alpha)$.

Since the application of the maximum principle is somewhat technical, we first present a special case to illustrate the idea. Consider Floer trajectories in the symplectisation $\mathbb{R} \times M$ and a time-independent Hamiltonian $H = 0$ which vanishes globally. In this case, also $X_H = 0$ and the Floer equation simplifies to $\partial_t u = J(u)\, \partial_s$.

Proposition 3.63. *Let $J \in \mathcal{J}(M, \alpha)$ and suppose $u = (f, v): \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times M$ satisfies Floer's equation with $J_t \equiv J$ and $H \equiv 0$. Then the function $f: \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ has no local maximum.*

Proof. We will show that f is subharmonic, i.e. that its Laplacian is non-negative, and apply the maximum principle. (We can apply Corollary 3.60 since $\mathbb{R}\times\mathbb{S}^1$ is biholomorphic to $\mathbb{C}\setminus\{0\}$; an explicit biholormorphic map is given by $\mathbb{R}\times\mathbb{S}^1 \ni (s,t) \mapsto e^{s+2\pi it} \in \mathbb{C}\setminus\{0\}$.)

By hypothesis, u satisfies the equation $\partial_t u = J(u)\,\partial_s u$. We want to write this in a more useful form, comparing the derivatives of f with those of v. To that end, observe that the tangent spaces of the symplectisation have a natural splitting

$$T_{(t,p)}(\mathbb{R}\times M) = \mathbb{R}\,\partial_r \oplus \mathbb{R}\,R_\alpha(p) \oplus \xi_p.$$

We consider the canonical projections

$$\pi_1 \colon T(\mathbb{R}\times M) \to \mathbb{R}\,\partial_r, \qquad \pi_2 \colon T(\mathbb{R}\times M) \to \mathbb{R}\,R_\alpha, \qquad \pi_\alpha \colon T(\mathbb{R}\times M) \to \xi.$$

We claim that the Floer equation is equivalent to the three equations

$$\partial_s f - \alpha(\partial_t v) = 0, \qquad \partial_t f + \alpha(\partial_s v) = 0, \qquad \text{and} \qquad \pi_\alpha(\partial_s u) + J(u)\pi_\alpha(\partial_t u) = 0.$$

We show this by considering the projections π_1, π_2 and π_α separately. Since the almost complex structure J is cylindrical, we have

$$J(\partial_t f\ \partial_r) = \partial_t f\ R_\alpha, \qquad J(R_\alpha) = -\partial_r \qquad \text{and} \qquad J(\xi) = \xi.$$

Hence, $\pi_1(J(u)\,\partial_t u)) = -\partial_t \tilde{v}\ \partial_r$, thus $\xi = \ker\alpha$ implies

$$\begin{aligned} 0 &= dr(\pi_1(0)) = dr\big(\pi_1(\partial_s u)\big) + dr\big(\pi_1(J(u)\partial_t u)\big) \\ &= dr(\partial_s f\ \partial_r) + dr(\partial_t \tilde{v}\,R_\alpha) = dr(\partial_s f\ \partial_r) - \alpha(\partial_t v) \\ &= \partial_s f - \alpha(\partial_t v), \end{aligned}$$

where $\tilde{v}$ is the R_α-component of v. Observe that $\partial_t \tilde{v} = \alpha(\partial_t v)$ since $\xi = \ker\alpha$.
For the projection π_2, we have

$$\pi_2(J(u)\,\partial_t u) = \partial_t f\ R_\alpha \qquad \text{and} \qquad \pi_2(\partial_s u) = \partial_t \tilde{v}\ R_\alpha = \alpha(\partial_t v)\,R_\alpha,$$

hence we deduce

$$0 = \alpha(\pi_2(0)) = \partial_t f + \alpha(\partial_t v).$$

For the third equation, we just observe $\pi_\alpha(J(u)\,\partial_t u) = J(u)(\pi_\alpha \partial_t u)$ since is cylindrical, hence preserves ξ. This shows the desired equivalence.

Now we observe that since $d\alpha$ vanishes on R_α and $d\alpha(\cdot, J(\cdot))$ defines a bundle metric on ξ, the third equation implies

$$d\alpha(\partial_s u, \partial_t u) = d\alpha(\pi_\alpha \partial_s u, \pi_\alpha \partial_t u) = d\alpha(\pi_\alpha \partial_s u, J(u)\pi_\alpha \partial_s u) \geq 0,$$

with equality if and only if $\pi_\alpha \partial_s u = 0$. Thus, we compute

$$0 \leq d\alpha(\partial_s u, \partial_t u) = (\partial_s v)\big(\alpha(\partial_t v)\big) - (\partial_t v)\big(\alpha(\partial_s v)\big) = \partial_s(\partial_s f) + \partial_t(\partial_t f) = (\partial_s^2 + \partial_t^2) f,$$

and f is indeed subharmonic. In the second step, we used the Lichnerowicz rule

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]);$$

the Lie bracket vanishes since ∂_s and ∂_t are coordinate tangent vectors. The third step uses the first two equations and a short computation along the lines of $(\partial_t)_p(\partial_t f) = (\partial_t v)_{v(p)}(\alpha(\partial_t v))$ for $p \in \mathbb{R} \times \mathbb{S}^1$. □

This result will imply a uniform C^0-bound for Floer trajectories $u \in \mathcal{M}(x,y,H,J)$. Note that there is no lower bound on the $\mathbb{R}$-component of u in the symplectisation. Such a bound is not needed since the negative end of the symplectisation will be replaced by a Liouville domain.

Let us now turn to the general case. We want to ensure suitable energy and C^0-bounds for Floer trajectories on $\widehat{W}$. In view of Proposition 3.63, the correct condition on J is to be admissible (see Definition 3.62). Let us see which conditions we need for the Hamiltonians. At the end of the day, we need to obtain C^0-bounds for the s-dependent Floer equation

$$\partial_s u + (J_{s,t} \circ u)\left(\partial_t u - X_{H_{s,t}}(u)\right) = 0.$$

Here, $\{J_{s,t}\}_{s,t\in\mathbb{R}\times\mathbb{S}^1}$ is a smooth family of almost complex structures in $\mathcal{J}(W,\omega,\alpha)$ that are s-independent for $|s| \geq C$ for some given[10] constant $C > 0$, so for $T_1 \geq T_0$ sufficiently large we can assume all of them to be in $\mathcal{J}(M,\alpha)$ on $[T_1,\infty) \times M$. Likewise, suppose $\{H_{s,t}\}_{s,t\in\mathbb{R}\times\mathbb{S}^1} \colon \widehat{W} \to \mathbb{R}$ is a smooth family of Hamiltonians which are s-independent for $|s| \geq C$. In light of Proposition 3.53, we assume them to take the form $H(s,t,r,p) = h(s,t,e^r)$ for $(r,p) \in [T_1,\infty) \times M$, with h being a smooth function on $\mathbb{R} \times \mathbb{S}^1 \times [T_1,\infty)$. For $T \geq T_0$, denote the complement of $(T,\infty)\times M$ by $\widehat{W}_T$, so that $\widehat{W} = \widehat{W}_T \sqcup (T,\infty) \times M$.

Proposition 3.64. *Given the data above, assume $\partial_s\partial_r h$ is everywhere non-negative. Let $x, y \in \mathcal{P}(H)$ be Hamiltonian orbits and u be a solution of the s-dependent Floer equation (Fs) which is asymptotic to x resp. y. For $T \geq T_1$, if x and y are contained in $\widehat{W}_T$, then also* $\operatorname{im}(u) \subset \widehat{W}_T$.

Proof. Suppose $u \colon \mathbb{R} \times \mathbb{S}^1 \to \widehat{W}$ satisfies the s-dependent Floer equation (Fs) and $U \subset \mathbb{R} \times \mathbb{S}^1$ is an open subset with $u(U) \subset [T_1,\infty) \times M$. Hence, for $(s,t) \in U$ we can write $u(s,t) = (f(s,t), v(s,t)) \in \mathbb{R} \times M$.

Again, we will show that f satisfies a maximum principle. Proposition 3.63 showed this for the case $H_{s,t} \equiv 0$ and $J_{s,t}$ being independent of s and t. We perform the same argument again. A simple computation analogous to Lemma 3.50 shows that $X_{H_{s,t}}(r,p) = (\partial_r h)(s,t,e^r) R_\alpha(p)$ for all $r \geq T_1$. Using that, we compute that $u = (f,v) \colon U \to \widehat{W}$ satisfies the equations

$$\partial_s f - \alpha(\partial_t v) - \partial_r h(s,t,e^f) = 0, \quad \partial_t f + \alpha(\partial_s v) = 0, \text{ and } \quad \pi_\alpha(\partial_s u) + J_{s,t}(u)\pi_\alpha(\partial_t u) = 0.$$

[10] The s-dependent Floer equation will come in when considering moduli spaces $\mathcal{M}(x,y,\{H_s\},\{J_s\})$ for a given homotopy $\{H_{s,t}, J_{s,t}\}$ between regular pairs. The constant C depends on the homotopy, hence is global within each moduli space.

Just as before, we have $0 \leq d\alpha(\pi_\alpha \partial_s u, J_{s,t}\pi_\alpha \partial_s u) = d\alpha(\partial_s u, \partial_t u)$, hence repeating the same calculation yields

$$\begin{aligned}
0 \leq d\alpha(\partial_s u, \partial_t u) &= (\partial_s v)(\alpha(\partial_t v)) - (\partial_t v)(\alpha(\partial_s v)) \\
&= \partial_s(\partial_s f) + \partial_t(\partial_t f) - \partial_s\left[\partial_r h(s,t,e^{f(s,t)})\right] \\
&= (\partial_s^2 + \partial_t^2) f - \partial_s\, \partial_r h(s,t,e^f) - e^f \partial_r^2 h(s,t,e^f)\, \partial_s f.
\end{aligned}$$

Since $\partial_s\, \partial_r h(s,t,e^f) \geq 0$ by hypothesis, f satisfies the partial differential inequality

$$(\partial_s^2 + \partial_t^2) f - e^f \partial_r^2 h(s,t,e^f)\, \partial_s f \geq 0.$$

The left hand side defines an elliptic differential operator, hence we can apply Proposition 3.59 locally.

This argument shows that f has no local maximum. If x and y are not contained in $[T_0, \infty) \times M$, since u is asymptotic to x and y we obtain that U must be compact, and empty by the above. Hence, $\mathrm{im}(u)$ is even contained in $\widehat{W}_{T_1}$. If x and y are contained in $[T_0, \infty) \times M$, for the $\mathbb{R}$-coordinate r_x resp. r_y of x resp. y, we have $\lim_{t \to -\infty} f(s,t) = r_x$ and $\lim_{t \to \infty} f(s,t) = r_y$. Since f has no local maximum, we obtain $f(s,t) \leq \max(r_x, r_y)$ for all s, t. $\square$

This reveals another property which is needed to define continuation maps in this setting: if the Hamiltonians $H_{s,t}$ satisfy[11] $H(s,t,r,p) = H(s,t,e^r)$ on some cylindrical end $[T, \infty) \times M$, the condition $\partial_s \partial_r h(s,t,e^r) \geq 0$ must hold, i.e. their slopes $\partial_r h(s,t,e^r)$ on this end should get steeper under the homotopy. With this assumption, C^0-bounds are ensured by Proposition 3.64, and energy bounds follow from Equation (3.2), which is still valid in this setting.

This maximum principle clears the biggest obstacle towards defining a Floer homology in this setting. We now turn towards the definition of the homology. In analogy to Hamiltonian Floer homology, we would like there to be only finitely many Hamiltonian orbits, i.e. we need a suitable non-degeneracy condition on the Hamiltonians.

In light of Proposition 3.53, this can only be true if the slope $\partial_r h(e^r)$ of the Hamiltonian $H(r,p) \approx h(e^r)$ on the cylindrical end $[T_0, \infty) \times M$ is not a period of a Reeb orbit of α. A priori, it is not clear whether this is possible. Luckily, if one allows a small perturbation of the contact form, the set of periods of Reeb orbits is discrete (see below).

Recalling Proposition 3.9, we should strive for a non-degeneracy condition of the Reeb vector field. Since the Reeb vector field is time-independent, every Reeb orbit γ comes in a family parametrised by $\mathbb{S}^1$: any orbit $\gamma(\cdot + t_0)$ is also an orbit of R_α. In particular, Reeb orbits are never non-degenerate in the sense of Definition 3.8. The next best condition is the following.

[11] Again, this condition should hold only up to a suitable small perturbation.

Definition 3.65. *Let X be a smooth vector field on a smooth manifold M. Denote the set of periodic orbits of X by $\mathcal{P}$. Recall that a T-periodic orbit $x \in \mathcal{P}$ of X is called* non-degenerate *if and only if the flow ϕ_t of X satisfies $\det(d\phi_T(x(0)) - \mathrm{id}) \neq 0$.*
It is called transversely non-degenerate *if and only if near x, the set $\mathcal{N} := \{y(0) \colon y \in \mathcal{P}\}$ is a 1-dimensional submanifold of M such that $\ker(d\phi_T(y(0)) - \mathrm{id}) = T_{y(0)}\mathcal{N}$ for all $y \in \mathcal{P}$ near x.*

Remark 3.66. The literature usually contains a different definition, that the linearisation $d\phi_T|_{\xi_{x(0)}} \colon \xi_{x(0)} \to T_{x(T)}M = T_{x(0)}M$ has no eigenvalue 1. Since the tangent space $T_{y(0)}N$ is the kernel of $d\phi_T$, this is equivalent to our definition if the submanifold N is 1-dimensional. We chose the definition above since it generalises to higher-dimensional submanifolds N.

Being transversely non-degenerate is a generic property: every contact form for ξ admits a small perturbation which makes it transversely non-degenerate.

Proposition 3.67 ([ABW10, Theorem A.1])**.** *Let $(M, \xi = \ker\alpha)$ be a contact manifold with contact form α. There exists a comeagre subset $\Lambda_{reg} \subset \{f \in C^\infty(M) \mid f > 0\}$ such that for each $f \in \Lambda_{reg}$, every periodic orbit of $R_{f\alpha}$ is transversely non-degenerate.* $\square$

Hence, using Remark 2.62, we may assume that α is transversely non-degenerate. This will suffice for our purposes.

Definition 3.68. *If α is a smooth contact form on a smooth manifold M, the set* $\mathrm{Spec}(M, \alpha) \subset (0, \infty)$ *of periods of the orbits of R_α is called the* action spectrum.

Non-degeneracy of the contact form is extremely helpful.

Proposition 3.69. *If α is transversely non-degenerate contact form on a compact smooth manifold M, then* $\mathrm{Spec}(M, \alpha)$ *is discrete and bounded away from* 0.

Proof outline. Since the Reeb vector field is non-zero and M is compact, there is a global lower bound for the periods of Reeb orbits. This follows from compactness of M and the local normal form for R_α (see e.g. [Lee02, Theorem 17.12]).

Discreteness of the spectrum is an application of the Arzelà-Ascoli theorem: consider an infinite sequence (γ_n) of Reeb orbits of periods $T_n \leq T$, reparametrised to yield maps $\gamma_n \colon \mathbb{S}^1 \to M$. Since their periods are bounded, there is a uniform C^∞-bound for the maps γ_n. (The C^0-bound follows from compactness of M, the C^1-bound holds since $\gamma_n'(t) = T_n R_\alpha(\gamma(t))$ and the periods T_n are bounded; higher derivatives follow easily.) Since M is compact, the Arzelà-Ascoli theorem yields a subsequence (γ_{n_k}) which converges to a limiting map $\gamma \colon \mathbb{S}^1 \to M$. The lower bound above excludes the case that γ is constant; hence γ is a Reeb orbit of period at most T. The convergence $\gamma_{n_k} \to \gamma$ implies that γ is not transversely non-degenerate, contradiction. $\square$

Remark 3.70. The spectrum of M is generally unbounded: if γ is a Reeb orbit, going through γ multiple times is also one, hence M has Reeb orbits of unbounded periods. In general, one should also expect distinct orbits of unbounded periods. Note however, that the *existence* of a Reeb orbit is non-trivial: the *Weinstein conjecture* states that any contact manifold $(M, \xi = \ker \alpha)$ has at least one Reeb orbit for R_α; this conjecture is still open in general.

Now, we are ready to formulate the full conditions necessary to define a theory of symplectic homology. By a small perturbation if necessary, assume that α is a transversely non-degenerate contact form for $(M, \xi = \ker \alpha)$. Let $T_0 > 0$ be such that $\widehat{W}$ contains the cylindrical end $([T_0, \infty) \times M, d(e^r \alpha))$.

Definition 3.71. *Let $\tau > 0$ with $\tau \notin \mathrm{Spec}(M, \alpha)$. A smooth Hamiltonian $H \colon \mathbb{S}^1 \times \widehat{W} \to \mathbb{R}$ is called* τ-admissible *if and only if*

- *for some $T \geq T_0$ and some $c \in \mathbb{R}$, we have $H_t(r, p) = \tau\, e^r + c$ on $[T, \infty) \times M$ (up to a small perturbation), and*
- *every 1-periodic orbit of X_{H_t} is non-degenerate.*

H is called admissible *if and only if it is τ-admissible for some $\tau \notin \mathrm{Spec}(M, \alpha)$. We denote the space of all admissible Hamiltonians by $\mathcal{H}(W, \omega)$.*

Since $\tau \notin \mathrm{Spec}(M, \alpha)$ and an admissible Hamiltonian H has constant slope at infinity, H has *no* 1-periodic orbits on $[T, \infty) \times M$, hence there are only finitely many Hamiltonians orbits in total.

Remark 3.72. In fact, we could consider a larger set of Hamiltonians, where the first condition would be weakened to assuming that for some $T \geq T_0$, we have $H(t, r, p) = h(e^r)$ on $[T, \infty) \times M$ (up to perturbation) such that the set $\{r \in \mathbb{R} \colon \partial_r(h(e^r)) \in \mathrm{Spec}(M, \alpha)\}$ is discrete. For such a Hamiltonian H and a regular $J \in \mathcal{J}(W, \omega, \alpha)$ (see Theorem 3.75 below), one will still obtain a well-defined Floer homology $SH(H, J)$.

We still have to think about an additional technical detail, namely the completeness of the Hamiltonian vector field. However, this is automatic for admissible Hamiltonians.

Observation 3.73. For $H \in \mathcal{H}(W, \omega)$, the Hamiltonian vector field X_{H_t} is complete.

Proof sketch. For Hamiltonians H with $H = h(e^r)$ on some cylindrical end, orbits in the cylindrical end are contained in level sets $\{r\} \times M$, which are compact. Since a Liouville domain is compact, orbits within it are also complete. Hence X_H is complete for such Hamiltonians. Since any sufficiently small perturbation of a complete vector field is complete, the same holds for any admissible Hamiltonian $H \in \mathcal{H}(W, \omega)$. $\square$

For a pair of an admissible almost complex structure and admissible Hamiltonian, we can perform the construction in Section 3.1 mutatis mutandis. The contractible loop space can be defined just as before. Since $\omega = d\lambda$ is exact, one can define the symplectic action function without recourse to a spanning disc.

Definition 3.74. *For an exact symplectic manifold $(M, \omega = d\lambda)$ and a Hamiltonian $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$, the symplectic action functional $\mathcal{A}_H\colon \Omega_0(M) \to \mathbb{R}$ is given by*

$$A_H(\gamma) := +\int_{\mathbb{S}^1} \gamma^*\lambda - \int_{\mathbb{S}^1} H(t, \gamma(t))\,\mathrm{d}t.$$

Again, we consider the formal positive gradient flow of $\mathcal{A}_H$, which yields the familiar Floer equation. Any critical point still has a well-defined Conley-Zehnder index (since $c_1(W, \omega) = 0$ by hypothesis); we define the Floer complex in the same way as before. The corresponding transversality theorem in our setting is the following.

Theorem 3.75 ([Oan08; FHS95])**.** *For an admissible Hamiltonian $H \in \mathcal{H}(W, \omega)$, there is a comeagre subset $\mathcal{J}^{reg} \subset \mathcal{J}(W, \omega, \alpha)$ such that for all $J \in \mathcal{J}^{reg}$, the moduli space*

$$\mathcal{M}(x, y, H, J) := \Big\{ u \in C^\infty(\mathbb{R} \times \mathbb{S}^1, M)\colon u \text{ solves (F), } u \text{ is contractible,} \\ \lim_{s\to\infty} u(s,t) = x(t) \text{ and } \lim_{s\to\infty} u(s,t) = y(t) \Big\}$$

is a smooth manifold of dimension $\mu_{CZ}(y) - \mu_{CZ}(x)$. The natural $\mathbb{R}$-action $s\cdot u = u(\cdot + s, \cdot)$ acts freely and properly on the space $\mathcal{M}(x, y, H, J)$, hence the quotient $\mathcal{M}(x, y; H, J)/\mathbb{R}$ is a smooth manifold of dimension $\mu_{CZ}(y) - \mu_{CZ}(x) - 1$. □

Again, we will call such a pair (H, J) a *regular pair* and from now on only consider regular pairs. By virtue of the assumptions we placed, we again obtain a compactness and gluing theorem. In the end, we obtain the following.

Proposition 3.76 ([Oan08; FHS95])**.** *For every regular pair (H, J) with $H \in \mathcal{H}(W, \omega)$ and $J \in \mathcal{J}(W, \omega, \alpha)$, there is a well-defined chain complex $(SC(H, J), \partial)$ defined by*

$$SC_k(H, J) = \bigoplus_{\substack{\gamma \in \mathcal{P}(H) \\ \mu(\gamma) = k}} \mathbb{Z}\langle\gamma\rangle$$

whose differential $\partial\colon SC_*(H, J) \to SC_{*-1}(H, J)$ satisfies

$$\partial(\langle y\rangle) = \sum_{\substack{x \in \mathcal{P}(H) \\ \dim \mathcal{M}(x,y,H,J) = 1}} \#\mathcal{M}(x, y, H, J)/\mathbb{R}\ \langle x\rangle,$$

where $\#$ denotes a count of points with signs as determined by a system of coherent orientations. Its homology is denoted by $SH(H, J) := H_*(SC(H, J), \partial)$. □

Remark 3.77. We can also define symplectic homology over any coefficient ring R. We denote the symplectic homology with R-coefficients by $SH(H, J; R)$. Most of the time, we will suppress the ring R from the notation.

Note that the chain complex $SC(H, J)$ only depends on the Hamiltonian H, while the differential depends on both H and J. The homology $SH(H, J)$ will *not* be independent of the data (H, J), nor should we expect it to be, since the Hamiltonian orbits of a τ-admissible Hamiltonian detect only Reeb orbits of period up to τ. (The homology *will*, however, be independent of the almost complex structure J.) Fortunately, the maximum principle (Proposition 3.64) allows us to define continuation maps between regular pairs. Generally, they will only go in one direction (and not both), since the maximum principle required an additional condition on the slopes of the Hamiltonians at infinity. Yet, we can still take the direct limit over these pairs, which *will* be an invariant. This is the symplectic homology we wanted to define.

Let us make this precise. To define the direct limit, we first need a preorder on the set of regular pairs. We do this via the admissible Hamiltonians.

Lemma/Definition 3.78. *We define a preorder $\leq$ on $\mathcal{H}(W, \omega)$ as follows: we define $H \leq H'$ if and only if there exist a constant $C \in \mathbb{R}$ and a compact set K such that $H(t, x) \leq H'(t, x) + C$ for all $x \in \widehat{W} \setminus K$. For two regular pairs (H_1, J_1) and (H_2, J_2), we define $(H_1, J_1) \leq (H_2, J_2) :\Leftrightarrow H_1 \leq H_2$.*

Remark 3.79. Equivalently, if H_1 and H_2 are τ-admissible and τ'-admissible, respectively, we have $H_1 \leq H_2 \Leftrightarrow \tau \leq \tau'$. Hence, $\mathcal{H}(W, \omega)$ with this preorder is a directed set. Note that the preorder is not antisymmetric, hence not a total order.

This definition is useful: if $H_1 \leq H_2$, there exists a continuation map for the associated Floer homologies. To prove this, one constructs a homotopy from H_1 to H_2 with increasing slope at the cylindrical ends and applies the maximum principle (Proposition 3.64) to obtain C^0 bounds. From that point, the same machinery as in Section 3.1 applies.

Proposition 3.80 (e.g. [Gut14, Definition 1.2.6])**.** *Let (H_1, J_1) and (H_2, J_2) be two regular pairs of admissible data, and suppose $H_1 \leq H_2$. Then there is a well-defined map $\phi\colon SH(H_1, J_1) \to SH(H_2, J_2)$.* □

We call these maps *continuation maps* again. In the same way as in Section 3.1, one shows that the maps ϕ are independent of the chosen homotopy. Moreover, they also satisfy the analogous composition laws.

Proposition 3.81 (e.g. [Gut14, Theorem 1.2.9])**.** *Given three regular pairs (H_1, J_1), (H_2, J_2) and (H_3, J_3) of admissible data with $H_1 \leq H_2$ and $H_2 \leq H_3$, the induced continuation maps $\phi_{12}\colon SH(H_1, J_1) \to SH(H_2, J_2)$, $\phi_{23}\colon SH(H_2, J_2) \to SH(H_3, J_3)$ and $\phi_{13}\colon SH(H_1, J_1) \to SH(H_3, J_3)$ satisfy the relation*

$$\phi_{23} \circ \phi_{12} = \phi_{13},$$

and the maps $\phi_{ii}\colon SH(H_i, J_i) \to SH(H_i, J_i)$ are the identity map for $i = 1, 2, 3$. □

Corollary 3.82. *If $H \in \mathcal{H}(W, \omega)$ and $J, J' \in \mathcal{J}(W, \omega, \alpha)$ such that (H, J) and (H, J') are regular pairs, then $SH(H, J)$ and $SH(H, J')$ are isomorphic.*

Proof. Since $H \leq H$, there are continuation maps $\phi\colon SH(H,J) \to SH(H,J')$ and $\psi\colon SH(H,J') \to SH(H,J)$ which are mutually inverse. $\square$

The continuation maps define a directed system, and we can take the direct limit.

Definition 3.83. *If R is any commutative ring, the* symplectic homology *of the Liouville domain (W,ω) is defined as the direct limit*

$$SH(W,\omega,X;R) := \varinjlim SH(H,J;R),$$

over all regular pairs (H,J), with the partial order and continuation maps from above.

Remark 3.84. This discussion extends extends to the larger class of Hamiltonians considered in Remark 3.72: for a regular pair (H,J), the Floer homology $SH(H,J)$ is well-defined, Definition 3.78 applies verbatim and we obtain continuation maps the same way. The symplectic homology is again defined as the direct limit of the system of continuation maps.

Remark 3.85. The attentive reader will remember that this definition incurred a choice of the contact form α on $M = \partial W$; the question is whether $SH(W,\omega;R)$ depends on this choice. While the space $\mathcal{J}(W,\omega,\alpha)$ of admissible almost complex structures clearly depends on this choice, the completion $(\widehat{W},\widehat{\omega})$ does not (up to symplectomorphism), hence such a dependence would be surprising. Indeed, one can show that the choice does not matter, by constructing continuation maps for homotopies of J between $\mathcal{J}(W,\omega,\alpha)$ and $\mathcal{J}(W,\omega,\alpha')$ for any two contact forms α and α' [Sei08, Section 3e].

The symplectic homology of a Liouville domain W is an invariant under exact symplectomorphism of its completion. This is *almost* obvious: all data used to define the complex $SC(H,J)$ depends only on the symplectic form ω, the Hamiltonian H and almost complex structure J. (While the symplectic actional functional technically depends on the primitive, by Stokes' theorem using spanning discs as in Definition 3.6 is equivalent. The latter definition involves only ω and is well-defined since exact symplectic manifolds are symplectically aspherical. The key part is that the symplectic forms are exact, and this exactness is preserved by exact symplectomorphisms.) The differential, partial order and continuation maps only depend on ω, hence $SH(W,\omega)$ is invariant under exact symplectomorphisms. However, this argument misses the subtle point that Definition 3.83 also depends on *the domain W*.

More generally, it is natural to ask whether symplectic homology is invariant under Liouville homotopies. By Lemma 2.67 and Proposition 2.70, a homotopy of Liouville domains induces an exact symplectomorphism of the completions. Hence, this reduces to the question above, and a rigorous proof would be doubly useful. Fortunately, there is one; we refer the reader to Seidel's paper [Sei08, Section 3e].

Theorem 3.86 ([Sei08, p. 13])**.** *If W and W' are Liouville isomorphic Liouville domains, their symplectic homologies $SH(W)$ and $SH(W')$ are isomorphic.* $\square$

While Definition 3.83 involves a finite type Liouville manifold, it does depend on the Liouville domain that is completed, hence is an invariant of that domain. However, we can extend the definition to a finite type Liouville domain M: if W is any Liouville domain whose completion is exact symplectomorphic to a M, we define $SH(M) := SH(W)$. This is well-defined since Liouville isomorphic Liouville domains have invariant symplectic homology, and is invariant under exact symplectomorphism by definition.

One can even extend the definition to general Liouville manifolds, by using an exhaustion by Liouville domains, verifying that the Viterbo transfer map yields maps $SH(V) \to SH(V_k)$ and taking the direct limit; this will be an invariant under Liouville homotopies and exact symplectomorphisms as well. We refer the reader to either Seidel's paper [Sei08, Section 7] or [CE12, Section 17.1] for the details.

The reader may wonder whether the use of a direct limit is necessary to compute the symplectic homology. In our setting, it is not. Consider a non-degenerate Hamiltonian $H_\infty\colon \mathbb{S}^1 \times \widehat{W} \to \mathbb{R}$ such that on $[T_0, \infty) \times M$, one has $H_\infty(t, r, p) = h(e^r)$ up to perturbation, where h is a smooth function such that $\partial_r h(e^r)$ is strictly increasing and which satisfies

$$0 < \tau = \partial_r h(e^{T_0}) < \inf\{\tau > 0\colon \tau \in \operatorname{Spec}(M, \alpha)\}, \qquad \text{and} \qquad \lim_{t\to\infty} \partial_r h(e^t) = \infty.$$

The Hamiltonian H_∞ belongs to the class in Remark 3.72, hence one can choose $J \in \mathcal{J}(W, \omega, \alpha)$ such that (H_∞, J) is a regular pair; hence $SH(H_\infty, J)$ is well-defined (and independent of J by the same argument as for Corollary 3.82). For each admissible Hamiltonian $H \in \mathcal{H}(W, \omega)$, we have $H \leq H_\infty$, hence there is a well-defined continuation map $\phi_H\colon SH(H, J) \to SH(H_\infty, J)$. The maps ϕ_H commute with the continuation maps for the admissible Hamiltonians, hence yield a map $\phi_\infty\colon SH(W, \omega) \to SH(H_\infty, J)$. In our setting, the map ϕ_∞ is an isomorphism [Sei08]. This does not mean, however, that we can easily dispense with the direct limit construction: the easiest proof that $SH(H_\infty, J)$ is independent of the choice of H_∞ works by using the direct limit.

Even computing with a single Hamiltonian can be difficult, since we have no control over the Hamiltonian orbits below the cylindrical end $[T_0, \infty) \times M$. This can be improved by considering a smaller class of Hamiltonians. Since the Hamiltonian H_∞ is just a singleton set which is cofinal, the appropriate generalisation is to consider a cofinal set. Eventually, whether to compute the symplectic homology via a single Hamiltonian H_∞ or a direct limit over a suitable cofinal set is a matter of taste.

Definition 3.87. *Let $\epsilon = \frac{1}{2}\inf\{\tau > 0\colon \tau \in \operatorname{Spec}(M, \alpha)\}$ be half the smallest period of a Reeb orbit in (M, α).*[12] *A smooth non-degenerate Hamiltonian $H\colon \mathbb{S}^1 \times \widehat{W} \to \mathbb{R}$ is called* good *if and only if there exist some $T \geq T_0$ and a convex increasing smooth function $h\colon [0, e^T] \to \mathbb{R}$ such that the following three properties are satisfied.*

*(1) H is **admissible** (w.r.t. the choice of T above): for some $c \in \mathbb{R}$ and some $\tau \notin \operatorname{Spec}(M, \alpha)$, we have $H_t(r, p) = \tau e^r + c$ on $[T, \infty) \times M$.*

[12] It is possible to choose ϵ differently; we make this choice in this document.

(2) *H is C^2-**small** on $\mathbb{S}^1 \times W$: for $(t,x) \in \mathbb{S}^1 \times W$, we have $|H(t,x)| < \epsilon$.*

(3) *$H(t,r,p)$ is C^2-**close to** $h(e^r)$ on $\mathbb{S}^1 \times [0,T] \times M$: we have $h(e^r) < \epsilon/2$ for $r \in [0,T]$, and $|\mathcal{A}_H(\gamma) - \mathcal{A}_{h(e^r)}(\gamma)| < \epsilon/2$ holds for any loop γ in $\mathbb{S}^1 \times [0,T] \times M$.*

Perhaps first off, we should convince ourselves that these Hamiltonians can actually be used to compute the symplectic homology. By Proposition 3.43, it suffices to show that they form a cofinal subset with respect to the order in Definition 3.78.

Proposition 3.88. *The subset of good Hamiltonians forms a cofinal subset of $\mathcal{H}(W,\omega)$.*

Proof sketch. It suffices to show that for each $\tau \notin \operatorname{Spec}(M,\alpha)$, there is a good Hamiltonian which is τ-admissible; this implies the statement by Remark 3.79. For each τ, it is easy to write down a time-independent Hamiltonian H_0 which satisfies items (1) through (3). A generic perturbation of H_0 will be non-degenerate; and every sufficiently small perturbation supported on $(\mathbb{S}^1 \times W) \cup (\mathbb{S}^1 \times [0,T] \times M)$ preserves the properties (1)–(3). □

In Section 3.4, we will explain that these conditions drastically simplify the description of the Hamiltonian orbits, and see that they allow us to define a further invariant called *positive symplectic homology*.

3.4. Positive symplectic homology

In this section, we will understand how the class of good Hamiltonians (see Definition 3.87) naturally gives rise to an invariant called *positive symplectic homology*. We keep the same setup as in Section 3.3.

The first insight is that for good Hamiltonians, 1-periodic Hamiltonian orbits come in precisely two kinds which can be distinguished by their action. We will need the following simple computation.

Lemma 3.89. *Let M be a smooth manifold, λ a smooth 1-form on M and X a smooth vector field on M. Let $\gamma\colon [0,T] \to M$ be a periodic orbit of X. Then*

$$\int_{[0,T]} \gamma^*\lambda = \int_0^T \lambda(X) = \int_0^T \lambda(X_{\gamma(t)})\,\mathrm{d}t. \qquad \square$$

Proposition 3.90. *Let $\epsilon = \frac{1}{2}\inf\{\tau > 0\colon \tau \in \operatorname{Spec}(M,\alpha)\}$. The 1-periodic orbits of a good admissible Hamiltonian $H \in \mathcal{H}(W,\omega)$ fall into two kinds:*

(A) *constant orbits at critical points in W, with action less than ϵ*

(B) *non-constant orbits in $\widehat{W} \setminus W$, with action greater than ϵ*

Proof. Condition (1) in Definition 3.87 implies that the only 1-periodic orbits of X_H in W are constant orbits at a critical point of H. By condition (1), their symplectic action is $-\int_0^1 H_t(x)\,\mathrm{d}t < \epsilon$.

Since H is admissible, there is no Hamiltonian orbit in some cylindrical end $[T,\infty)\times M$. Hence, all non-constant Hamiltonian orbits run in $[0,T)\times M \subset \widehat{W}\setminus W$. Let us estimate their action. Let τ be the slope of H at infinity, i.e. H is τ-admissible, and consider a non-constant orbit x of X_H in $[0,T)\times M$. By condition (3) in Definition 3.87, there exists a closed Reeb orbit of period $T<\tau$ such that x is close to this orbit located in the level set $\{r\}\times M$ with $T = h'(e^r)$. Hence, the action of x is close to the action of the orbit γ of $h'(e^r)R_\alpha$. We want to compute the latter, which by definition is

$$A_H(\gamma) = +\int_{\mathbb{S}^1}\gamma^*\lambda - \int_{\mathbb{S}^1} H(t,\gamma(t))\,\mathrm{d}t.$$

For the first term, we have $\lambda = e^r\alpha$ on $\{r\}\times M$; since γ is an orbit of $h'(e^r)R_\alpha$, by Lemma 3.89 above we have

$$\int_{\mathbb{S}^1}\gamma^*\lambda = \int_{\mathbb{S}^1} e^r\alpha(h'(e^r)R_\alpha)\,\mathrm{d}t = \int_{\mathbb{S}^1} e^r h'(e^r)\,\mathrm{d}t = e^r h'(e^r) = e^r T.$$

For the second term, since the Reeb orbit γ is contained in a level set $\{r\}\times M$ and approximately $H(r,p) = h(e^r)$, we have $H(t,\gamma(t)) \approx h(e^r)$, hence we have

$$\int_{\mathbb{S}^1} H(t,\gamma(t))\,\mathrm{d}t \approx \int_{\mathbb{S}^1} h(e^r)\,\mathrm{d}t = h(e^r).$$

Hence, we compute $\mathcal{A}_H(\gamma) \approx e^r h'(r) - h(e^r)$. By condition (2) in Definition 3.87, we have $h(e^r) < \epsilon/2$; by our choice for ϵ we have $T \geq 2\epsilon$. Hence, the action $A_h(\gamma) = e^r T - h(e^r)$ is greater than $3/2\epsilon$. Thus, by condition (3) in Definition 3.87, the action $\mathcal{A}_H(\gamma_H)$ of γ_H is greater than ϵ. □

Note that in the second case, we have $\partial_r(e^r h'(r) - h(e^r)) = e^r h''(r) > 0$ since h is convex, hence the action is increasing with r. In other words, every level in the symplectisation corresponds to a specific period of Reeb orbits.

For the rest of this section, we only consider good admissible Hamiltonians H. In light of Proposition 3.90, it is natural to define an invariant by quotienting out one kind of orbits. In order to still have a well-defined subcomplex, the differential should descend to the quotient, and we have to pay attention to the direction of the action.

From Section 3.1, we recall that the symplectic action *increases* along Floer trajectories: if $u\colon \mathbb{R}\times\mathbb{S}^1 \to \widehat{W}$ solves Floer's equation (F) such that $\lim_{s\to-\infty} u(s,t) = x(t)$, $\lim_{s\to\infty} u(s,t) = y(t)$, then $\mathcal{A}_H(x) \leq \mathcal{A}_H(y)$. However, in the definition of the chain complex (see Proposition 3.76), such a trajectory contributes a summand mapping $\langle y\rangle$ to (plus or minus) $\langle x\rangle$, hence maps the orbit y to another orbit x of *smaller* action. Hence, we make the following definition. Again, let R be a commutative ring with unit.

Definition 3.91. *Let $SC^{\leq\epsilon}(H, J; R)$ be the chain complex generated by the 1-periodic orbits of H whose action is at most ϵ. Since the differential decreases the action of the Hamiltonian orbits, the differential descends to this complex, hence this is a well-defined subcomplex of $SC(H, J; R)$. By Proposition 3.90, it is built of critical points of H.*

Theorem 3.92 ([Vit99, Proposition 1.3])**.** *$H_*(SC^{\leq\epsilon}(H, J; R) \cong H_{n-*}(W, \partial W; R)$.* □

Definition 3.93. *The* positive Floer complex *of (W, ω) is defined by quotienting out the subcomplex of critical points: $SC^+(H, J; R) := SC(H, J; R)/SC^{\leq\epsilon}(H, J; R)$. Since the differential ∂ decreases the action of the critical points, the differential descends to $SC^+(H, J; R)$, hence the homology $SH^+(H, J; R) := H_*(SC^+(H, J; R), \partial)$ is a well-defined R-module.*

The continuation maps for the complexes $SH(H, J; R)$ descend to continuation maps for the $SH^+(H, J; R)$, since continuation maps also decrease the action. Hence, we can take the direct limit of the $SH^+(H, J; R)$ over the set of good admissible Hamiltonians.

Definition 3.94. *The* positive symplectic homology *of (W, ω) is defined as*

$$SH^+(W, \omega, X; R) := \varinjlim H_*(SC^+(H, J; R), \partial).$$

When R is understood, we will often suppress it from notation; similarly for ω and X.

Remark 3.95. The heuristic argument before Theorem 3.86 applies here just as well and suggests that positive symplectic homology is invariant under exact symplectomorphisms (the action functional is preserved by exact symplectomorphisms, as argued there). To the contrary, whether positive symplectic homology is invariant under Liouville homotopies is much less apparent, since a Liouville homotopy could vary the action functional and hence the subcomplexes $SC^{\leq\epsilon}(H, J)$ used to define positive symplectic homology.

Remark 3.96. By Proposition 3.90, $SH^+(W, \omega, X; R)$ is generated by the Reeb orbits on $(M = \partial W, \xi = \ker\alpha)$. In view of Remark 3.70, we see that $SH^+(W, \omega, X; R)$ is infinitely generated in general. This is a pronounced difference to singular homology, compare Proposition 6.3.

Remark 3.97. Similar to Remark 3.85, the positive symplectic homology is independent of the choice of contact form α. Note that as an R-module, each chain complex $SC^+(H, J; R)$ depends only on $(M, \xi = \ker\alpha)$ since it is generated by non-constant Hamiltonian orbits in the cylindrical end $\widehat{W} \setminus W$ which correspond to Reeb orbits in $(M, \xi = \ker\alpha)$. On the other hand, the differential for $SC^+(H, J; R)$ may depend on the filling W of (M, ξ) since a Floer trajectory between non-constant Hamiltonian orbits might go into the filling. Thus, different Liouville fillings of (M, ξ) may have different positive symplectic homology.

Let us give an example where this happens.

Example 3.98 ([Laz17, Example 2.8])**.** We consider two different Liouville fillings of $(\mathbb{S}^1, \xi_{\mathrm{std}})$. Observe that for all $g \geq 0$, the once-punctured genus g surface Σ_g (i.e. a closed surface of genus g with a small disc removed) is a Weinstein filling of $\mathbb{S}^1$.

Since $\Sigma_0 = \mathbb{C}$ is subcritical (with the standard Weinstein structure, there is precisely one critical point, of index 0), we have $SH(\Sigma_0) = 0$, hence the long exact sequence (Proposition 3.99 below) will imply $SH^+(\Sigma_0) \cong H(\Sigma_0, \mathbb{Z}) \cong \mathbb{Z}$. On the other hand, for $g \geq 1$, the homology $SH(\Sigma_g)$ is infinite-dimensional [Sei08, Example 3.3], hence $SH^+(\Sigma_g)$ is infinite-dimensional for $g \geq 1$ by the long exact sequence. (By Corollary 2.127, the singular homology is always finite-dimensional, hence cannot compensate for the infinite-dimensional $SH^+(\Sigma_g)$.) Thus, $SH^+(\Sigma_g)$ is different for $g = 0$ and $g = 1$.

In Section 5.1, we will discuss a class of contact manifolds (M, ξ) for which $SH^+(W)$ is independent of the filling W of (M, ξ).

3.5. Properties of symplectic homology

We can now elaborate on a few key properties of symplectic homology which we will need later. Let R be a commutative ring with unit. In our thesis, the most important property will be a long exact sequence.

Proposition 3.99. *Let (W, ω, X) be a Liouville domain with $c_1(W) = 0$, let $(\widehat{W}, \widehat{\omega})$ be its completion and R a commutative ring with unit. There is a long exact sequence*

$$\cdots \to SH^+_{*+1}(W, \omega, X; R) \to H_{n-*}(W, \partial W; R) \to SH_*(W, \omega, X; R) \to SH^+_*(W, \omega, X; R) \to \cdots,$$

where H_* denotes singular homology.

Proof. By definition, we have a short exact sequence of chain complexes

$$0 \to SC^{\leq \epsilon}(H, J; R) \to SC(H, J; R) \to SC^+(H, J; R) \to 0,$$

which induces a long exact sequence of the graded homology R-modules

$$\cdots \to H_{n-*}(W, \partial W; R) \to SH_*(H, J; R) \to SH^+_*(H, J; R) \to H_{n-*+1}(W, \partial W; R) \to SH_{*-1}(H, J; R) \cdots$$

By Theorem 3.47, taking the direct limit of each term in the sequence still yields a long exact sequence. Hence, we obtain a long exact sequence.

$$\cdots \to SH^+_{*+1}(W, \omega, X; R) \to H_{n-*}(W, \partial W; R) \to SH_*(W, \omega, X; R) \to SH^+_*(W, \omega, X; R) \to \cdots$$

This is exactly the sequence we were looking for. $\square$

Secondly, the symplectic homology is not only a graded abelian group or R-module, but also admits a ring structure, with the multiplication given by the *pair-of-pants product*. We just outline the high-level idea. Instead of counting cylindrical solutions to the Floer equation (as one does for Hamiltonian Floer theory or symplectic homology), one counts solutions to a similar partial differential equation, defined on a sphere with three punctures (a "pair of pants"). More precisely, one considers a thrice punctured sphere Σ and maps $u\colon \Sigma \to \widehat{W}$ which are asymptotic to Hamiltonian orbits in $\widehat{W}$ near the punctures and satisfy a suitable partial differential equation. This equation is not as easy to state as Floer's equation; we just say that in a neighbourhood of each puncture, the equation looks like a Floer equation—but potentially with different Hamiltonians and almost complex structures around each puncture. Given two Hamiltonian orbits $\alpha, \beta \in \mathcal{P}(H)$ in $\widehat{W}$, one defines their product as

$$\langle\alpha\rangle \cdot \langle\beta\rangle := \sum_{\gamma \in \mathcal{P}(H)} \mu(\alpha, \beta; \gamma)\langle\gamma\rangle,$$

where $\mu(\alpha, \beta, \gamma)$ is a signed count of the number of "pair-of-pants maps" which are asymptotic to the orbits α, β and γ. Again, one has to study the possible compactness phenomena and analyse the corresponding moduli spaces to show that this is well-defined. In the end, one obtains a compatibility condition with the boundary map on the chain complexes: $\mu(\partial\alpha, \beta) \pm \mu(\alpha, \partial\beta) = \partial\mu(\alpha, \beta)$ for a suitable sign in the middle; hence this defines a well-defined product map on the product of the homologies. Finally, note that unlike the Floer equation (but similar to its continuation maps), the equation considered here is not invariant under an $\mathbb{R}$-action, hence the product is grading-preserving: the only non-zero summands are for orbits γ with $|\alpha| + |\beta| = |\gamma|$.

We refer the reader to either Matthias Schwarz' PhD thesis [Sch95] or Ritter's paper [Rit13] for details. For this document, the following result is sufficient.

Proposition 3.100 ([Rit13])**.** *Let (W, ω, X) be a Liouville domain with $c_1(W) = 0$, $(\widehat{W}, \widehat{\omega})$ its completion and R a commutative ring with unit. The pair-of-pants product $SH_*(W; R) \otimes SH_*(W; R) \to SH_{*+*-n}(W; R)$ makes $SH_*(W; R)$ into a unital ring, with the unit in $SH_n(W; R)$.* □

The singular homology has a ring structure: by Poincaré-Lefschetz duality, there is an isomorphism $H_{n-*}(W, \partial W; R) \to H^*(W; R)$ to singular cohomology; the latter has a ring structure with unit via the cup product. Thus, the map $H_{n-*}(W, \partial W; R) \to SH_*(W, \omega, X; R)$ from the exact sequence is a well-defined map between rings. In fact, it is also a unital ring homomorphism.

Proposition 3.101 ([Rit13, Theorem 6.6])**.** *Let R be a commutative unital ring. The morphism $H_{n-*}(W, \partial W; R) \to SH_*(W; R)$ in Proposition 3.99 is a unital ring homomorphism.* □

The third important property is the existence of the so-called *Viterbo transfer map*: if $V \subset W$ is a Liouville subdomain (see below), there is a natural map $SH(W) \to SH(V)$

[Vit99], which is compatible with the ring structure given by the pair-of-pants product. The precise definition of Liouville subdomains is the following.

Definition 3.102. *Let (W, ω, X) be a Liouville domain. A compact submanifold $V \subset W$ of codimension zero is called a* Liouville subdomain *of W if and only if the vector field X is pointing outwards along the boundary ∂V.*

Remark 3.103. Hence, a subset $V \subset W$ of a Liouville domain (W, ω, X) is a Liouville subdomain if and only if the induced exact symplectic structure yields a Liouville domain $(V, \omega|_V, X|_V)$ with $\dim V = \dim W$.

The precise statement of the Viterbo transfer map is the following.

Proposition 3.104 ([Vit99], [Rit13, Theorem 9.5])**.** *Let R be a unital commutative ring. Let W be a Liouville domain and $V \subset W$ a subdomain. There is a natural map $SH(W; R) \to SH(V; R)$, called* Viterbo transfer map*, which is both an R-module homomorphism and a unital ring homomorphism.* □

The transfer map is non-trivial because it is a unital ring homomorphism and the naturality property. More precisely, it satisfies the following.

Proposition 3.105 ([Vit99; CO18, Proposition 5.4])**.** *Let W be a Liouville domain, then the Viterbo transfer map $SH(W) \to SH(W)$ is the identity map. If $U \subset V \subset W$ are Liouville subdomains, the corresponding transfer maps $f_{UV} \colon SH(V) \to SH(U)$, $f_{VW} \colon SH(W) \to SH(V)$ and $f_{UW} \colon SH(W) \to SH(U)$ satisfy $f_{UW} = f_{UV} \circ f_{VW}$.* □

Gutt extended this construction to positive symplectic homology [Gut14; Gut17]. The transfer map is again natural.

Proposition 3.106 ([Gut17])**.** *Let W be a Liouville domain and $V \subset W$ a subdomain. For each unital commutative ring R, there is a well-defined R-module homomorphism $SH^+(W; R) \to SH^+(V; R)$.* □

Proposition 3.107 ([Gut14, Theorem 3.1.12; CO18, Proposition 5.4])**.**
Let W be a Liouville domain. The transfer map $SH^+(W) \to SH^+(W)$ is the identity map. If $U \subset V \subset W$ are Liouville subdomains, the corresponding transfer maps $f_{UV} \colon SH^+(V) \to SH^+(U)$, $f_{VW} \colon SH^+(W) \to SH^+(V)$ and $f_{UW} \colon SH^+(W) \to SH^+(U)$ satisfy $f_{UW} = f_{UV} \circ f_{VW}$. □

Let us explicitly note the following consequence.

Corollary 3.108. *If $V \subset W$ is a Liouville subdomain, then $SH(W) = 0$ implies $SH(V) = 0$.*

Proof. Consider the Viterbo transfer map $\phi \colon SH(W) \to SH(V)$. Since ϕ is a unital ring homomorphism, $SH(W) = 0$ implies $SH(V) = 0$. □

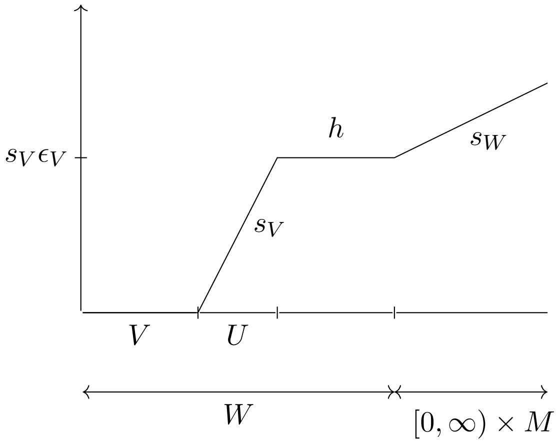


Figure 3.1.: Sketch of a function h as used for the construction of the Viterbo transfer map. Manually reproduced based on [Laz17, Figure 2].

We omit proofs for the ring homomorphism and naturality statements. Yet, let us outline the construction of the Viterbo transfer maps; details can be found e.g. in [Laz17, Section 2.5]. Let $V \subset (W, \lambda_W)$ be a Liouville subdomain. Hence, there is a collar U of $(Z, \alpha_Z) = (\partial V, \lambda_V|_{\partial V})$ in $W \setminus V$ such that $(U, \omega|_W)$ is symplectomorphic to $(Z \times [1, 1+\epsilon_V], d(e^r\alpha_Z))$. Let $(M, \alpha_M) = (\partial W, \lambda_W|_{\partial W})$. The first step is to establish a suitable version of the maximum principle. With the same proof as for Proposition 3.64, one can show the following.

Proposition 3.109 ([Laz17, Lemma 2.5])**.** *Let $V \subset (W, \lambda_W)$ be a Liouville subdomain, let $(Z, \alpha_Z) = (\partial V, \lambda_V|_{\partial V})$, U be a collar as above and $\widehat{W}$ be the completion of W. Consider $H\colon \widehat{W} \to \mathbb{R}$ such that $H = h(e^r)$ is increasing near Z. Let $J \in \mathcal{J}(W, \omega, \alpha)$ be admissible and also cylindrical in U, i.e. $J|_U$ preserves $\xi_Z = \ker \alpha_Z$, $J|_{\xi_Z}$ is independent of t and $J(\partial_t) = R_{\alpha_Z}$. If both asymptotic orbits of a (H, J)-Floer trajectory $u\colon \mathbb{R} \times \mathbb{S}^1 \to \widehat{W}$ are contained in V, then u is contained in V.* $\square$

Then, the proof considers a suitable cofinal subset of Hamiltonians, those which are a small perturbation of Hamiltonians $H(t, r, p) = h(e^r)$ for a smooth function $h\colon (0, \infty) \to \mathbb{R}$ with the following properties.

- $h \equiv 0$ on V
- h is linear on U with slope s_V
- $h \equiv s_V\epsilon_V$ in $W \setminus (V \cup U)$
- h is linear with slope s_W on $\widehat{W} \setminus W = [0, \infty) \times M$, where $s_V, s_W \notin \operatorname{Spec}(Z, \alpha_Z) \cup \operatorname{Spec}(M, \alpha_M)$

One shows that Hamiltonian orbits for such H fall into several classes, similar to Proposition 3.90. The set of orbits in U, and those which remain in $W \setminus (V \cup U)$ are invariant under the differential, hence form a subcomplex of each chain complex $SH(V, H, J)$ resp. $SH(W, H, J)$. Quotienting by these and taking the direct limit, one obtains the desired transfer maps $SH(W) \to SH(V)$ and $SH^+(W) \to SH^+(V)$.

We close this chapter with the following observation. With Corollary 5.17 in Section 5.2, we will follow up on this result.

Observation 3.110. If W and W' are Liouville domains with $c_1(W) = 0 = c_1(W')$, and $\widehat{W}$, $\widehat{W'}$ their completions, the disjoint union $W \sqcup W'$ is again a Liouville domain with completion $\widehat{W} \sqcup \widehat{W'}$. For any commutative unital ring R, there is an isomorphism $SH_*(W \sqcup W; R) \to SH_*(W; R) \oplus SH_*(W'; R)$.

Proof. Since there are no Floer trajectories between orbits in W and W', the chain complex for $W \sqcup W'$ is the direct sum of the chain complexes for W and W'. $\square$

4. Main result, related work and strategy of the proof

In this chapter, we state Zhou's main theorem, give an overview of previous research on the topic and outline Zhou's strategy of proof.

Research towards uniqueness of symplectic fillings goes back to the end of the 80s. The first result (by combining work of Gromov [Gro85] and Eliashberg [Eli90]) shows that a Liouville filling of the standard contact 3-sphere $(\mathbb{S}^3, \xi_{\text{std}})$ is unique up to symplectic (or even Weinstein) deformation. There have been a number of further results in dimension four since then. However, in higher dimensions such results up to symplectic deformation are notably missing. The main reason is that dimension four allows to apply *intersection theory*: in a four-dimensional manifold, two generic complex curves intersect in finitely many points; their *intersection number* is a signed count of these intersection points with multiplicities. Conditions on the intersection number give useful information about the behaviour of such curves; this makes it possible to control the symplectic fillings via such curves.

The first result valid for all dimensions was by Eliashberg, Floer and McDuff [McD91, Theorem 1.5]. However, their result governs only the diffeomorphism type of the filling.

Theorem 4.1 (Eliashberg-Floer-McDuff [McD91]). *Let (W, ω) be a symplectically aspherical strong symplectic filling of $(\mathbb{S}^{2n-1}, \xi_{std})$, $n \geq 3$. Then W is diffeomorphic to the ball $\mathbb{D}^{2n}$.* □

We will not define strong fillings, but note that any Liouville filling is a strong symplectic filling. Recall that Liouville fillings are symplectically aspherical (Corollary 3.55). Their result was generalised by Barth, Geiges and Zehmisch [BGZ16, Theorem 1.5].

Theorem 4.2 (Barth-Geiges-Zehmisch [BGZ16]). *If (M, ξ) is a simply connected contact manifold which has a subcritical Stein filling, all symplectically aspherical strong fillings of (M, ξ) are diffeomorphic.* □

Again, this result only concerns the diffeomorphism type of fillings.[1] Regarding the hypotheses, we remark that Stein fillings are equivalent to Weinstein fillings.

[1] While [BGZ16] also have a result about symplectomorphism type, this is merely an immediate corollary of the flexibility properties of Weinstein domains.

The first result in higher dimensions about the symplectomorphism type of fillings was in a paper of Seidel, attributed to Smith [Sei08, Corollary 6.5]. In our terminology, they show the following.[2]

Theorem 4.3 (Smith, [Sei08]). *Any Liouville filling W of the standard contact sphere $(\mathbb{S}^{2n-1}, \xi_{std})$ with $n \geq 2$ has $SH(W) = 0$.* □

Essential progress towards generalising this theorem was made in a preprint by Lazarev [Laz17, Corollary 4.2]. Zhou's paper [Zho18] improves his result: in Lazarev, the statement that the filling W has vanishing symplectic homology is a part of the hypothesis; Zhou makes this a part of the conclusion. A prototypical version of Zhou's result is the following. (We will state and prove the full version in Chapter 6.)

Theorem 4.4 ([Zho18, Corollary 1.3]). *Let (M, ξ) be a contact manifold with* $\dim M \geq 5$ *and $c_1(\xi) = 0$ that has a flexible Weinstein filling W. Then any topologically simple Liouville filling W' of (M, ξ) satisfies $SH_*(W'; \mathbb{Z}) = 0$ and $H^*(W; \mathbb{Z}) \cong H^*(W'; \mathbb{Z})$.* □

As Zhou explains in his introduction, the Eliashberg-Floer-McDuff theorem implies that any Liouville filling of $(\mathbb{S}^{2n-1}, \xi_{\mathrm{std}})$ is topologically simple, hence Zhou's theorem generalises Smith's result.

Since both proofs share several common features, we will elaborate on the strategy of proof. We will see that while Smith-Seidel's approach generalises, their methods do not.

The case $n = 2$ in Smith' proof is answered by a theorem by Gromov, that any Liouville filling of the standard contact sphere $\mathbb{S}^3$ is Liouville isomorphic to the standard symplectic ball $\mathbb{D}^4$ [Gro85]. For $n \geq 3$, the proof strategy consists of three elements:

(1) Firstly, they show that for any Liouville filling M, each symplectic homology group $SH_k(M)$ has rank at most one.

(2) Secondly, they observe that for any Liouville filling M, the boundary connected sum $M\#M$ (see Definition 2.118) is another topologically simple Liouville filling of $(\mathbb{S}^{2n-1}, \xi_{\mathrm{std}})$. Hence, its symplectic homology also satisfies these rank bounds.

(3) The final step is a grading-preserving isomorphism $SH(M\#M) \to SH(M) \oplus SH(M)$ provided by the Viterbo transfer map (see Corollary 5.17). Hence, if some $SH_k(M)$ had rank 1, then $SH_k(M\#M)$ had rank 2, which is a contradiction. Consequently, the symplectic homology must be trivial, which completes the proof.

In the first step, Smith-Seidel use the Eliashberg-Floer-McDuff theorem to conclude that M is diffeomorphic to $\mathbb{D}^{2n}$. Hence, both the cohomology of the filling M and its boundary $\mathbb{S}^{2n-1}$ are standard computations; there are two generators for $\mathbb{S}^{2n-1}$ and one for M.

[2] Strictly speaking, their result is about symplectic *co*homology, not homology. However, these are related via $SH^*(W) = SH_{n-*}(W)$ [Zho18, Remark 2.11], hence one vanishes if and only if the other does.

Next, they define a Morse-Bott spectral sequence approximating the symplectic homology of M. Smith-Seidel are very short on details here; the interested reader is recommended to read Kwon and von Koert's paper [Kv16, Section 5.2] to learn about the construction. The spectral sequence's first page contains vanishing terms and the cohomologies of M and the sphere [Sei08, Equation (3.2)]. The cohomology of the sphere has just two generators and the filling has one, which happen to be positioned so that each diagonal in the first page contains at most one term. The theory of spectral sequences then implies that the subsequent pages will have the same property, hence the rank bound follows and the first step is complete.

For the second step, they use that the boundary connected sum of two fillings is a filling of the contact connected sum (see Proposition 2.119) and the general fact that the contact connected sum of $(\mathbb{S}^3, \xi_{\mathrm{std}})$ with itself yields the standard contact 3-sphere again.

Zhou's proof uses the same three-step strategy as Smith-Seidel. However, their methods cannot be used directly in this context. The spectral sequence argument requires precise topological information about the filling as input, but one could use the result by Barth, Geiges and Zehmisch in analogy to the Eliashberg-Floer-McDuff theorem. This applies in a more restrictive setting that Zhou's methods, but could work in principle.

To the contrary, defining a Morse-Bott spectral sequence is an issue. In general, this sequence can be defined only in a Morse-Bott setting (in which the problem has extra symmetries)—this is true for the sphere since it has a free $\mathbb{S}^1$-action given by rotation, but not for arbitrary contact manifolds. Moreover, one cannot expect the topology of a general contact manifold or Liouville filling to be as simple as for the standard sphere: one can construct Weinstein domains by successive handle attachment, hence the topology in the subcritical dimensions can get rather complicated. Consequently, a different strategy is needed.

Instead, Zhou uses the exact sequence and ring structure of symplectic homology. The first ingredient is that a flexible Weinstein filling necessarily has vanishing symplectic homology (Theorem 6.1). The second key ingredient is the insight that for a suitable class of contact manifolds, the positive symplectic homology $SH^+(W)$ is independent of the choice of topologically simple Liouville filling W. Combining these insights with the exact sequence shows that the $(n+1)$-st symplectic homology has rank at most one, and applying the connected sum argument above shows vanishing in that level. The vanishing of all symplectic homology groups can be deduced using the ring structure and the exact sequence again. The details will be presented in Chapter 6.

Hence, Zhou's proof relies on two crucial ingredients: it works for a class of contact manifolds for which (1) the positive symplectic homology is independent of the choice of topologically simple Liouville filling, and (2) which is invariant under boundary connected sums. Statements for property (1) are not new in the literature; this property was already known for all *dynamically convex* contact manifolds (see Definition 5.3) by results of Bourgeois and Oancea [BO16]. However, this class of manifolds is not invariant under boundary connected sums [Laz17, p. 40]. Lazarev [Laz17] managed to find a more general class of contact manifolds which satisfy both conditions, namely *asymptotically dynamically convex* contact manifolds (see Definition 5.8), of which flexible Weinstein fillable contact manifolds are an example (see Proposition 5.15). With this class of manifolds, the strategy above can be carried out and yields a proof of the main theorem.

In Chapter 5, we explain and motivate the definition of asymptotically dynamically convex manifolds, sketch how the first property is proven and make the second one precise. The result proof of Zhou's theorem is described in Chapter 6.

5. Asymptotically dynamically convex manifolds and their key properties

As we already mentioned, the key technical concept to making Zhou's proof work is the class of asymptotically dynamically convex contact manifolds. In this chapter, we explain their definition and outline two of their key properties. In fact, the definition becomes more transparent through the context of the first property: the positive symplectic homology of Liouville fillings of a given contact manifold is not unique in general, but being asymptotically dynamically convex is a sufficient condition to ensure that. The second property is that ADC contact manifolds are invariant under boundary connected sums. We will not explain the general proof, but sketch an argument in the special case of flexible Weinstein domains.

5.1. Positive symplectic homology independent of filling

Let us look at contact manifolds for which all topologically simple Liouville fillings have the same positive symplectic homology. This chapter essentially parallels Lazarev [Laz17, Sections 3.1 and 3.2]. We will work in the following setting.

Convention. Let $(M, \xi = \ker \alpha)$ be a $(2n-1)$-dimensional contact manifold with $c_1(M, \xi) = 0$.

For each contractible Reeb orbit γ w.r.t. α, one can define a Conley-Zehnder index similarly to Section 3.1: instead of the tangent bundle TM, one uses the symplectic vector bundle $(\xi, d\alpha) \to M$. One begins by choosing a trivialisation of TM along γ (which exists since M is orientable). In this trivialisation, the contact planes $\xi_{\gamma(t)}$ along γ define a path of symplectic matrices. The Conley-Zehnder index $\mu_{CZ}(\gamma) \in \mathbb{Z}$ of the Reeb orbit γ is obtained from this path in the same was in Section 3.1. We define the *degree* $|\gamma|$ of a Reeb orbit γ of R_α by the *reduced* Conley-Zehnder index $|\gamma| := \mu_{CZ}(\gamma) + n - 3$. We hint at the reason for this grading below.

The assumptions of contractibility and vanishing first Chern class are needed to make the Conley-Zehnder index well-defined. In order to define the Conley-Zehnder index, we need to assume that γ is non-degenerate; we can achieve this by perturbing the contact form α (see Proposition 3.67). For a general Reeb orbit γ, the index depends on the choice of trivialisation. For contractible orbits, any orbit extends to a map on an embedded disc $\mathbb{D}^2 \subset M$, and any choice of trivialisation of $TM|_{\mathbb{D}^2} = (\mathbb{D}^2)^*TM$ restricts to a trivialisation of γ^*TM. This choice does not matter: two different trivialisations of $TM|_{\mathbb{D}^2}$ induce homotopic trivialisations of γ^*TM. For two different embeddings, the resulting indices are equal by an argument similar to Proposition 3.10; this is where the assumption $c_1(M, \xi) = 0$ appears.

Finally, we note that if (M, ξ) has a Liouville filling (W, ω), we have $\iota^*c_1(W, \omega) = c_1(M, \xi)$ for the inclusion map $\iota\colon M \to W$. (This follows by applying Property 2.41(i) of the first Chern class to ι; i.e. $\iota^*\colon H^2(W) \to H^2(M)$ is the map in cohomology induced by ι.) Hence, $c_1(W, \omega) = 0$ implies $c_1(M, \xi) = 0$.

Next, we describe how the positive symplectic homology can be independent of the choice of Liouville filling. The first results for our purposes were shown by Bourgeois and Oancea [BO09; BO16]. We begin with outlining their argument. To simplify the discussion, let us pretend at first that all desired transversality results hold.

Proposition 5.1. *Let W be a Liouville filling of (M, ξ). If all Reeb orbits that are contractible in W have positive degree, the positive symplectic homology $SH^+(W)$ is independent of the filling W.*

Outline of proof. Recall from Section 3.4 that the chain modules $SC_k^+(H, J)$ depend only on (M, ξ), but the differential may depend on W, since it counts Floer trajectories which could go into W. Hence, proving invariance of $SH^+(W)$ requires some control over these Floer trajectories.

Bourgeois and Oancea use a procedure called *stretching the neck*, which we will explain on page 86, to change the almost complex structure J. For the resulting almost complex structure, they show that the Floer trajectories which enter W are in bijection with punctured Floer trajectories in the cylindrical end $(0, \infty) \times M$, capped off by "rigid J-holomorphic finite-energy planes" in $\widehat{W}$. Since the Floer trajectories in $(0, \infty) \times M$ are independent of W, the differential depends on W precisely through these J-holomorphic planes in $\widehat{W}$.

By their construction, these planes are asymptotic to Reeb orbits of $(M, \xi = \ker \alpha)$. Those orbits are contractible since they bound planes in W, hence have well-defined degree. If one assumes appropriate transversality results, the rigid J-holomorphic planes in $\widehat{W}$ have index[1] zero, and the orbits must have degree zero. (This is where the precise grading of the Reeb orbits comes in: it is chosen to match the "virtual dimension" of a suitable moduli space, meaning that—under suitable transversality results—a J-holomorphic disc is asymptotic to an orbit whose degree equals the disc's index. We don't have space to explain this in detail.)

[1] This index is not a Conley-Zehnder index, but a more general quantity called a *Fredholm index*. It occurs naturally in the setup of transversality results (including the cases mentioned in Chapter 3); if transversality results hold, the dimension of an appropriate moduli space is given by an appropriate Fredholm index.

However, by hypothesis, all orbits that are contractible in W have positive degree, hence there are no such J-holomorphic planes and all Floer trajectories stay in $\widehat{W} \setminus W = (0, \infty) \times M$. Hence, in this case the differential does not depend on the filling W and $SH^+(W)$ is an invariant of $(M, \xi = \ker \alpha)$. □

However, the criterion in this proposition depends on the filling, hence is not very helpful without knowing the filling. The solution is to consider π_1-injective fillings W of M, i.e. fillings for which the induced map $\iota_* \colon \pi_1(M) \to \pi_1(W)$ on the fundamental groups is injective. Hence, the following definition is natural.

Definition 5.2. *A Liouville domain W is called* topologically simple *if and only if $c_1(W) = 0$ and the map $\pi_1(\partial W) \to \pi_1(W)$ is injective. A Liouville filling W of a contact manifold M is called* topologically simple *if and only if W is a topologically simple Liouville domain.*

If W is π_1-injective, all Reeb orbits that are contractible in W are contractible in M, hence their degrees can also be computed in M. The same holds for all Hamiltonian orbits that are contractible in W (since they have the same trace by an analogue of Proposition 3.53), thus the grading of $SH^+(W)$ can also be determined from M directly. Hence, the necessary conditions for π_1-injective fillings are exactly described by the following definition.

Definition 5.3. *A contact manifold (M, ξ) is called* dynamically convex *if and only if there is a contact form α for ξ such that all contractible Reeb orbits of α have positive degree.*

Hence, one arrives at the following result. Let us note that the first result of this type is due to Eliashberg, Givental and Hofer [EGH00]. They considered contact structures which had no Reeb orbits of degree -1, 0 and 1, and showed a similar invariance result in the context of an invariant called "cylindrical contact homology".

Proposition 5.4 ([BO16; EGH00])**.** *If (M, ξ) is dynamically convex, all π_1-injective Liouville fillings W of (M, ξ) have isomorphic $SH^+(W)$.* □

Discussion 5.5. The attentive reader may wonder why we assume that all orbits have positive (instead of just non-zero) degree. This is related to the transversality results: if all desired transversality results hold, the rigid J-holomorphic planes in W have index zero, hence should be asymptotic to a Reeb orbit of degree zero, and the absence of such degree zero orbits suffices to show independence of $SH^+(W)$ of the filling W.

However, transversality results for J-holomorphic planes do not hold in general. Luckily, they *do* hold for Floer trajectories in the cylindrical end $(0, \infty) \times M$. If all Reeb orbits have positive degree, transversality of these Floer trajectories can be used to exclude the formation of J-holomorphic finite energy planes planes in W from Floer trajectories. The argument does *not* seem to work if all orbits have non-zero (rather than positive) degree.

The condition of dynamically convex contact structures is too restrictive for our purposes, since it requires precise control over the degrees of *all* Reeb orbits. For example, contact surgery (see Section 2.5) creates many orbits with arbitrarily large action, hence dynamically convex contact structures are e.g. not invariant under boundary connected sums [Laz17, p. 40]. In particular, we don't have complete control over all orbits in the contact boundary of a flexible Weinstein domain. However, it turns out that full control is not needed. The insight (first put forward by Bourgeois, Ekholm and Eliashberg [BEE12]) is that restricting to orbits with bounded action yields much better behaviour. They compute invariants in terms of orbits of action less than some bound, and let the bound go to infinity. Lazarev [Laz17] found an appropriate generalisation of dynamically convex manifolds, taking a similar approach.

Giving this generalised definition requires a few definitions.

Definition 5.6. *The* action *of a T-periodic Reeb orbit γ in M is defined as*

$$\mathcal{A}(\gamma) := \int_{[0,T]} \gamma^* \alpha.$$

Note that $\mathcal{A}(\gamma)$ is always positive and equals the period of γ. Hence, we could equivalently define the action spectrum of (M, α) as $\operatorname{Spec}(M, \alpha) = \{\mathcal{A}(\gamma) \colon \gamma \text{ Reeb orbit of } \alpha\}$. For any real number D, consider the set

$$\mathcal{P}^{\leq D}(M, \alpha) := \{\gamma \colon \gamma \text{ contractible Reeb orbit of } R_\alpha \text{ with action } \mathcal{A}_H(\gamma) < D\}.$$

Since α was non-degenerate, each such set is finite for D fixed (see Proposition 3.69).

Observe that a strict contactomorphism preserves the contact forms, the Reeb vector field, the Reeb orbits and even preserves their action. Hence, the following holds.

Observation 5.7. If $\phi \colon (M, \xi = \ker \alpha) \to (M', \xi' = \ker \alpha')$ is a strict contactmorphism, the sets $\mathcal{P}^{\leq D}(M, \alpha)$ and $\mathcal{P}^{\leq D}(M', \alpha')$ are in grading-preserving bijection. □

Notation. If α, α' are positively oriented contact forms for (M, ξ), there exists a unique $f \colon M \to \mathbb{R}^+$ such that $\alpha' = e^f \alpha$. We write $\alpha' \geq \alpha$ (resp. $\alpha' > \alpha$) if and only if $f \geq 0$ (resp. $f > 0$). Note that if $\alpha' \geq \alpha$ (resp. $\alpha' > \alpha$), for any diffeomorphism $\phi \colon M' \to M$, we have $\phi^* \alpha' \geq \phi^* \alpha$ resp. $\phi^* \alpha' > \phi^* \alpha$.
Now we state the proper relaxation of dynamical convexity.

Definition 5.8 ([Laz17, Definition 3.6])**.** *A contact manifold (M, ξ) is called* asymptotically dynamically convex *if and only if there exist a sequence of non-increasing contact forms $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \ldots$ for ξ and increasing positive numbers $D_1 < D_2 < D_3 < \ldots$ going to infinity such that all elements of $\mathcal{P}^{\leq D_k}(M, \alpha_k)$ have positive degree.*

We will occasionally abbreviate asymptotically dynamically convex by ADC.

Even though asymptotically dynamically convex contact structures are more general than dynamically convex ones, Proposition 5.4 still holds for ADC contact structures.

Proposition 5.9 ([Laz17, Proposition 3.8])**.** *Let (M,ξ) be an asymptotically dynamically convex contact manifold. If W and W' are two topologically simple Liouville fillings of (M,ξ), then $SH^+(W;R) \cong SH^+(W';R)$ for any commutative unital ring R.*

The full proof of this statement is rather technical; hence we will only outline the main ideas. The first ingredient is “stretching the neck”, which is a method of deforming an almost complex structure on $\widehat{W}$ along a hypersurface. The next step generalises and formalises the argument that after sufficient neck-stretching, all Floer trajectories used in the construction of SH^+ (i.e. Floer trajectories for the chain complexes $SC(H,J)$, for continuation maps and for homotopies of homotopies) will be contained in $\widehat{W} \setminus W$, hence are independent of W and of the filling. In the final step, Lazarev shows that this neck-stretching does not change SH^+. With some analytical effort and invoking the ADC property, he constructs a commutative grid involving suitable continuation resp. Viterbo transfer maps. He then shows that this grid encodes $SH^+(W)$ and is independent of the neck-stretching. We will omit the details.

Let us close this discussion by explaining the neck-stretching in some detail. Let $V \subset \widehat{W}$ be a Liouville subdomain with contact boundary (Z,α_Z). Consider a collar of Z in V which is symplectomorphic to $([-\delta,0]\times Z, d(e^r\alpha_Z))$ for some small $\delta > 0$. Let $J \in J(W,\omega,\alpha)$ be an admissible almost complex structure on $\widehat{W}$ which is also cylindrical on $[-\delta,0]\times Z$, set $J' := J|_{[-\delta,0]\times Z}$. For each $R \in (-\infty,-\delta)$, we can extend J' to a cylindrical almost complex structure on $[R,0]\times Z$, which we still denote by J'.

Let $\phi^R\colon [R,0] \to [-\delta,0]$ be any diffeomorphism whose derivative equals 1 near the boundary. We define a compatible almost complex structure J^R by

$$J^R = \begin{cases} (\phi^R \times \mathrm{id})_* J' & \text{on } [-\delta,0]\times Z \\ J & \text{outside } [-\delta,0]\times Z. \end{cases}$$

This is smooth since ϕ^R has derivative 1 near the boundary; it is still admissible (for a different symplectic form ω^R) since $J^R = J$ on $[0,\infty)\times M$, hence is still cylindrical, and compatibility follows since ϕ is orientation-preserving. Observe that J^R admits a larger collar, hence we say the “neck (of J) was stretched” (from δ to R).

In formal language, the second step of the proof looks like the following. This is the simplest version (for the differential on $SC(H,J)$); the versions for continuation maps and homotopies of homotopies are more technical, but not more difficult.

Proposition 5.10 ([Laz17, Proposition 3.10])**.** *Let $H \in \mathcal{H}(W,\omega)$ and $J \in \mathcal{J}(W,\omega,\alpha)$ be admissible. Furthermore, assume that on $[-\delta,0]\times Z \subset V$, we have $H \equiv 0$ and J is cylindrical. Suppose $\iota_*\colon \pi_1(Z) \to \pi_1(W)$ is injective and all elements of $\mathcal{P}^{\leq D}(Z,\alpha)$ have positive degree. Let y and x be Hamiltonian orbits of H in $\widehat{W}$. If $\mathcal{A}_H(y) - \mathcal{A}_H(x) < D$, there exists an $R_0 \in (-\infty,-\delta)$ such that for any $R \leq R_0$, all rigid (H,J^R)-Floer trajectories are contained in $\widehat{W} \setminus V$.* □

The proof of Proposition 5.10 proceeds by contradiction, assuming that there exist a sequence $R_k \to -\infty$ and (H, J^{R_k})-Floer cylinders $u_k \in \mathcal{M}(x, y; H, J^{R_k})$ such that each u_k is not contained in $\widehat{W} \setminus V$. Using a general compactness result for such Floer cylinders (the *SFT compactness theorem*, which generalises Theorem 3.19 from Section 3.1), Lazarev shows that the u_k must converge to a "broken Floer building" (which is the natural limiting object in the SFT compactness theorem). Since transversality results hold in that setting, the dimension of the associated moduli space is related to the degrees above; the degree assumption implies that the dimension is negative, contradiction.

5.2. Surgery invariance properties

Asymptotically dynamically convex manifolds also have special properties with respect to surgery. One special case that is easier to grasp concerns flexible Weinstein domains. In Section 2.5, we saw that the boundary connected sum of two Weinstein domains is again a Weinstein domain. It turns out that the boundary connected sum of *flexible* Weinstein domains is also flexible. This requires a stronger argument. The following argument was explained to me by Oleg Lazarev.

Proposition 5.11. *The boundary connected sum of two flexible Weinstein domains in dimension at least five is again a flexible Weinstein domain.*

Proof sketch. Consider two flexible Weinstein domains W_1, W_2 and Weinstein homotopies to explicitly flexible domains W_1^{flex} and W_2^{flex}. Then the boundary connected sum $W_1^{\text{flex}} \natural W_2^{\text{flex}}$ is still explicitly flexible: we can attach the 1-handle away from a loose chart for the loose Legendrians along which the critical critical points are attached. By Remark 2.120 the choice of 1-handle does not matter. The Weinstein homotopies from W_1 to W_1^{flex} and W_2 to W_2^{flex} yield a Weinstein homotopy from $W_1 \natural W_2$ to the explicitly flexible domain $W_1^{\text{flex}} \natural W_2^{\text{flex}}$ as the 1-handle was attached away from the loose charts. $\square$

Remark 5.12. This statement implicitly uses the invariance of flexible Weinstein domains under Weinstein homotopy, since we didn't specify how to perform the boundary connected sum.

This result extends to asymptotically dynamically convex contact manifolds. We will use the following result.

Proposition 5.13 ([Laz17, Theorem 3.15])**.** *If (M, ξ) and (M', ξ') are asymptotically dynamically convex contact manifolds of dimension at least five, their contact connected sum $M \# M'$ is asymptotically dynamically convex.* $\square$

This is a special case of the following more general result.

Proposition 5.14 (Lazarev, [Laz17, Theorems 3.15, 3.17 and 3.18])**.** *If (M, ξ) is an ADC contact manifold of dimension at least five and (M', ξ') is obtained by performing subcritical*[2] *or flexible surgery on (M, ξ), then (M', ξ') is also ADC.* □

The proof relies on the h-principles for loose Legendrians and flexible Weinstein domains. Since it is somewhat involved, we skip the details.

As a consequence, Lazarev shows the following.

Proposition 5.15 ([Laz17, Corollary 4.1])**.** *Every flexible Weinstein fillable contact manifold of dimension at least five is asymptotically dynamically convex.*

Proof idea. This statement follows from handlebody decomposition (Proposition 2.125) of Weinstein domains and the result above. Some care must be taken about the attachment of 2-handles; the details can be found in Lazarev's proof. □

The closing result of this chapter concerns the change of symplectic homology under these operations. Cieliebak has shown that symplectic homology is invariant under subcritical handle attachment. In particular, this applies to boundary connected sums.

Proposition 5.16 ([Cie02, Theorem 1.11; Fau19, Theorem 9])**.** *Let W be a Liouville domain with $c_1(W) = 0$, and W' be the result of applying subcritical surgery to W along ∂W. Then $SH(W') \cong SH(W)$.* □

Corollary 5.17. *Let W and W' be Liouville domains with $c_1(W) = c_1(W') = 0$. Let $\iota \colon W \sqcup W' \to W \natural W'$ denote the inclusion. Then, for any commutative unital ring R, the Viterbo transfer map $\iota^* \colon SH_*(W \natural W'; R) \to SH_*(W) \oplus SH_*(W')$ arising from the map ι is an isomorphism.*

Proof. Combine Proposition 5.16 with Observation 3.110. □

[2]Some extra conditions need to be satisfied for surgery along 2-handles, see [Laz17, Theorem 3.17].

6. Proof of the main theorem

In this chapter, we finally present the proof of Zhou's theorem. Let us recall its prototypical version that we encountered in Chapter 4.

Recall (Theorem 4.4)**.** Let (M,ξ) be a contact manifold with $\dim M \geq 5$ and $c_1(\xi) = 0$ that has a flexible Weinstein filling W. Then any topologically simple Liouville filling W' of (M,ξ) satisfies $SH_*(W';\mathbb{Z}) = 0$ and $H^*(W;\mathbb{Z}) \cong H^*(W';\mathbb{Z})$.

We will outline the proof of this prototypical version first, and then indicate how it can be generalised. The first instrumental result is the following.

Theorem 6.1 ([BEE12; MS])**.** *A flexible Weinstein domain W satisfies $SH(W) = 0$.*
Proof approaches. The first proof would follow from a result by Bourgeois, Ekholm and Eliashberg [BEE12]. Their paper gives only heuristic proofs; the necessary technical details are supposedly provided in a recent preprint by Ekholm [Ekh19]. Hence, there is no consensus yet as to whether this is sufficient.

The proof relies on the handlebody decomposition 2.125 of the Weinstein domain W: one can obtain W from the unit disc $\mathbb{D}^n$ by attaching finitely many handles. One can compute $SH(\mathbb{D}^n) = 0$ explicitly (see e.g. [Gut14, p. 42][1]); by Cieliebak's result the symplectic homology stays invariant under attaching subcritical handles (Proposition 5.16). For critical surgery, Bourgeois-Ekholm-Eliashberg suggest an exact sequence

$$\cdots \to SH_*(W') \to SH_*(W) \to LH_*^{H_0}(\Lambda) \to SH_{*-1}(W') \to \cdots,$$

where W and W' are the Weinstein domain before and after a critical handle was attached, and the term $LH_*^{H_0}(\Lambda)$ is a version of the *Legendrian contact homology* of the attaching sphere Λ of the critical handle. We will not define Legendrian contact homology here, partially because there is no rigorous general definition yet.
For explicitly flexible domains, each critical handle is attached along loose Legendrians, for which the Legendrian contact homology (heuristically) vanishes. Hence, vanishing of the symplectic homology is preserved by each handle attachment. For flexible domains, $SH(W) = 0$ follows since the symplectic homology is invariant under Weinstein homotopy (by Theorem 3.86).

An alternative proof was given by Murphy and Siegel [MS, Theorem 3.2]. They use an h-principle for symplectic embeddings of flexible Weinstein domains to show that a flexible Weinstein domain symplectically embeds into a Liouville domain V for which $SH(V) = 0$ is known, and apply Corollary 3.108. □

[1] Note that in Gutt's conventions, the Hamiltonian vector field has the opposite sign.

Recall that every flexible Weinstein fillable contact manifold is asymptotically dynamically convex (by Proposition 5.15). In order to apply Proposition 5.9 about invariance of SH^+ of the filling, we furthermore must show that flexible Weinstein domains are topologically simple. We begin with two straightforward topological results.

Proposition 6.2. *Let W and W' be two topologically simple Liouville domains. Their boundary connected sum $W \natural W'$ is also topologically simple.*

Proof. By definition, $W \natural W'$ is obtained by adding a 1-handle to the disjoint union $W \sqcup W'$. Let $\iota \colon W \sqcup W' \to W \natural W'$ denote the inclusion. Since the disc $\mathbb{D}^{2n-1}$ and the 1-handle $\mathbb{D}^1 \times \mathbb{D}^{2n-1}$ both have trivial second cohomology group, the induced map $i_* \colon H^2(W \natural W'; \mathbb{Z}) \to H^2(W; \mathbb{Z}) \oplus H^2(W'; \mathbb{Z})$ is an isomorphism. Since the first Chern class is compatible with direct sums, we have $c_1(W \sqcup W') = c_1(W) + c_1(W') = 0$. Since $\iota^* c_1(W \natural W') = c_1(W \sqcup W')$, we have $c_1(W \natural W') = 0$.

To compute $\pi_1(W \natural W')$, we apply van Kampen's theorem. Since the 1-handle that is attached to W and W' is contractible, we just obtain a free product of groups, i.e. $\pi_1(W \natural W') \cong \pi_1(W) * \pi_1(W')$. Similarly, we compute $\pi_1(\partial W \# \partial W') \cong \pi_1(\partial W) * \pi_1(\partial W')$. Since both maps $\pi_1(\partial W) \to \pi_1(W)$ and $\pi_1(\partial W') \to \pi_1(W')$ are injective by hypothesis, so is the induced map on the free product. Hence, $W \natural W'$ is also topologically simple. $\square$

Proposition 6.3. *If W is a Liouville domain, the singular homology $H_*(W; \mathbb{Z})$ is finitely generated.*

Proof. This is the same idea as Corollary 2.127. Choose any Morse function $\phi \colon W \to \mathbb{R}$ (there always exists one [Mil63, Corollary 6.7]); since W is compact, ϕ has only finitely many critical points. Then use Proposition 2.122. $\square$

Now we can prove that all Weinstein fillings are topologically simple. In the proof, we employ the *relative homotopy groups* and their long exact sequence. See Hatcher's textbook [Hat02, Chapter 4] for their definition and properties.

Proposition 6.4. *Assume (W, λ, X) is a Weinstein filling of a contact manifold (M, ξ) with $\dim M = 2n - 1 \geq 5$ and $c_1(\xi) = 0$. Then W is topologically simple.*

Proof. By Corollary 2.126, the filling W can be constructed from the standard ball by attaching handles of index at most n. Equivalently, W can be constructed from M by attaching handles of index at least n. This implies $\pi_2(W, M) = 0$, where $\pi_2(W, M)$ is the *relative second fundamental group* of (W, M). By the homotopy long exact sequence of the pair (W, M), there is an exact sequence

$$\cdots \to \pi_2(W, M) \to \pi_1(M) \to \pi_1(W) \to \cdots,$$

thus $\pi_2(W, M) = 0$ implies that the map $\pi_1(M) \to \pi_1(W)$ is injective.

Let $\iota \colon M \to W$ denote the canonical inclusion and let J be an almost complex structure on W. By Property 2.41(i) we have $c_1(W, J) = \iota^* c_1(M, \xi)$. Consider the long exact sequence in cohomology for the pair (W, M): $H^2(W, M; \mathbb{Z}) \to H^2(W; \mathbb{Z}) \xrightarrow{\iota^*} H^2(M; \mathbb{Z})$.

By Poincaré-Lefschetz duality, we have $H^2(W,M) \cong H_{2n-2}(W)$; the latter term vanishes by Corollary 2.127 since $2n-2 \geq n+1$ as $n \geq 3$. Hence, $\iota^* \colon H^2(W;\mathbb{Z}) \to H^2(M;\mathbb{Z})$ is injective and $c_1(W) = 0$ follows. □

Now we can prove the prototypical form of Zhou's result.

Proof of Theorem 4.4. We first show $SH(W;k) = 0$ for any field k. Recall that the chain complex defining $SH_*(W;\mathbb{Z})$ is a free abelian group generated by periodic Hamiltonian orbits. Applying the universal coefficient theorem [ES52, p. 161, G3] to the free chain complexes $SC_*(H,J;\mathbb{Z})$ and taking the direct limit, we obtain an exact sequence

$$0 \to SH_*(W;\mathbb{Z}) \otimes k \to SH_*(W;k) \to \mathrm{Tor}(SH_{*-1}(W;\mathbb{Z});k) \to 0. \tag{6.1}$$

This uses the exactness of the direct limit functor (Theorem 3.47), and also the fact that direct limits commute with the tensor functor (Lemma 3.48). By Theorem 6.1, we have $SH_*(W;\mathbb{Z}) = 0$. Hence, all terms but the middle one vanish and $SH_*(W;k) = 0$ follows for any field k.

The long exact sequence Proposition 3.99 yields an isomorphism $SH_*^+(W;k) \cong H_{n-*+1}(W;k)$.[2] Since (M,ξ) is asymptotically dynamically convex (by Proposition 5.15) and both W and W' are topologically simple, Proposition 5.9 and the previous relation yield an isomorphism

$$SH_*^+(W';k) \cong SH_*^+(W;k) \cong H_{n-*+1}(W;k). \tag{6.2}$$

Applying Proposition 3.99 to W', we obtain a long exact sequence

$$\cdots \to H_{-1}(W';k) \to SH_{n+1}(W';k) \to SH_{n+1}^+(W';k) \to H_0(W';k) \to SH_n(W';k) \to \cdots. \tag{6.3}$$

Since $H_{-1}(W';k) = 0$, we have $SH_{n+1}(W';k) \cong \ker(SH_{n+1}^+(W';k) \to H_0(W';k))$. By relation (6.2), $SH_{n+1}^+(W';k)$ is isomorphic to $H_0(W;k) \cong k$. Since k is a field, there are only two possibilities for the subring $SH_{n+1}(W';k) \subset k$, hence we have

$$SH_{n+1}(W';k) \cong 0 \qquad \text{or} \qquad SH_{n+1}(W';k) \cong k. \tag{6.4}$$

This completes the first step of the proof outline: we have a bound on the rank of $SH_{n+1}(W';k)$. Next, we rule out the case $SH_{n+1}(W';k) \cong k$ using the connected sum.

By Proposition 2.119 and Proposition 5.11, the boundary connected sum $W \natural W$ is a flexible Weinstein filling of the contact connected sum $M \# M$. By Proposition 6.2, the boundary connected sum $W' \natural W'$ is still a topologically simple Liouville filling of $M \# M$, hence the argument for Equation (6.4) also applies to $W' \natural W'$, and we conclude that

$$SH_{n+1}(W' \natural W';k) \cong 0 \qquad \text{or} \qquad SH_{n+1}(W' \natural W';k) \cong k.$$

[2] Since we are working with coefficients in a field, we can drop the boundary term: over a field, the singular homology and cohomology of the same rank are isomorphic.

On the other hand, by Corollary 5.17 there is an isomorphism of $SH(W'\natural W';k)$ to $SH(W';k)\oplus SH(W';k)$, thus $SH_{n+1}(W';k)\cong k$ would imply $SH_{n+1}(W'\natural W';k)\cong k\oplus k$, a contradiction. Therefore, the only possibility is $SH_{n+1}(W';k)=0$.

Applying this to the long exact sequence (6.3), we see that the map $SH^+_{n+1}(W';k)\to H_0(W';k)$ is injective. Since $H_0(W';k)\cong k$ is a field, the map is an isomorphism. Hence, (6.3) yields that $H_0(W';k)\to SH_n(W';k)$ is the zero map. By Proposition 3.101, the map $H_0(W')\to SH_n(W',k)$ sends the unit in $H_*(W';k)$ to the unit in $SH_*(W';k)$, hence the unit in $SH_*(W';k)$ is zero. This shows $SH_*(W';k)=0$ for any field k.

Assume now that the integral symplectic homology $SH(W';\mathbb{Z})$ does not vanish. By Proposition 6.3, $H_{n-*}(W';\mathbb{Z})$ is a finitely generated abelian group. Hence, $SH^+_*(W;\mathbb{Z})\cong SH^+_*(W';\mathbb{Z})$ is also finitely generated by Proposition 3.99. Since both $SH^+_*(W';\mathbb{Z})$ and $H_{n-*}(W';\mathbb{Z})$ are finitely generated, so is $SH_*(W';\mathbb{Z})$ by Proposition 3.99. Thus, by the classification of finitely generated abelian groups, $SH_*(W';\mathbb{Z})$ contains either a summand $\mathbb{Z}$ or a summand $\mathbb{Z}_m$ for some integer m. In either case, there exists a prime p such that $SH(W';\mathbb{Z})\otimes\mathbb{Z}_p\neq 0$. But by (6.1), this implies that $SH(W';\mathbb{Z}_p)\neq 0$, in contradiction to $SH(W';k)=0$ for the field $k=\mathbb{Z}_p$.

Finally, combining Proposition 5.9 with $SH(W;\mathbb{Z})=SH(W';\mathbb{Z})=0$ and the exact sequence, Proposition 3.99, yields an isomorphism

$$H^*(W;\mathbb{Z})\cong H_{n-*}(W,\partial W;\mathbb{Z})\cong SH^+_{*+1}(W;\mathbb{Z})\cong SH^+_{*+1}(W';\mathbb{Z})\cong H^*(W';\mathbb{Z}). \qquad \square$$

Inspecting the proof in detail, we observe that we didn't use the full power of W being flexible: we used Theorem 6.1, Proposition 5.15 and the fact that flexible Weinstein domains are closed under boundary connected sums. Since ADC contact manifolds are also closed under boundary connected sums, the argument also applies to them. More precisely, Zhou's full result is the following.

Theorem 6.5 ([Zho18, Theorem 1.1])**.** *Let (M,ξ) be an asymptotically dynamically convex contact manifold of dimension $2n-1\geq 5$. Suppose M has a topologically simple Liouville filling W with $SH_*(W)=0$. Then every topologically simple Liouville filling W' of M satisfies $SH_*(W';\mathbb{Z})=0$ and $H^*(W;\mathbb{Z})\cong H^*(W';\mathbb{Z})$.*

Proof. Let k be any field. Since $SH(W)=0$ by hypothesis, as in the proof above we have $SH(W;k)=0$. The proof of $SH_{n+1}(W';k)\cong 0$ or k follows in the same way, except that M being ADC and W being topologically simple are now part of the hypotheses (as opposed to a consequence of W being flexible Weinstein).

By Proposition 5.13, the contact connected sums $M\#M'$ is ADC with Liouville fillings $W\natural W$ and $W'\natural W'$. The boundary connected sum $W\natural W$ is topologically simple with vanishing symplectic homology by Proposition 6.2 combined with Corollary 5.17. Hence, the first paragraph applies again and shows $SH_{n+1}(W'\natural W';k)\cong 0$ or k.

The remaining argument works exactly like for Theorem 4.4: we deduce $SH_{n+1}(W';k)\cong 0$, use the ring property to show $SH(W';k)=0$ and then infer $SH(W';\mathbb{Z})=0$. $\square$

In the proof, we have seen a curious interplay between flexibility and rigidity: Zhou's theorem is essentially a result that topologically simple Liouville fillings behave rigidly, but the objects involved are inherently flexible and the proof depends on this in an essential way: the first key result specific to this setting was that a flexible Weinstein filling has trivial symplectic homology, which holds because flexible Weinstein domains involve loose Legendrians which satisfy an h-principle. The second core insight was that flexible Weinstein domains are invariant under certain *surgery* operations. The actual proof connecting these insights was then algebraic, but of course depended on the analytical and topological machinery behind it.

7. Conclusion and outlook

We just saw the proof of Zhou's result, that for a flexible Weinstein fillable asymptotically dynamically convex (ADC) contact manifold of dimension at least five, any topologically simple Liouville filling has vanishing symplectic homology. This proof generalised almost verbatim to all ADC contact manifolds which admit a topologically simple Liouville filling W with $SH(W) = 0$. A key ingredient was that for ADC contact manifolds, all topologically simple Liouville fillings have the same SH^+. In a recent preprint, Zhou showed that this invariance of SH^+ for ADC manifolds is part of a larger picture [Zho19]. In addition to the positive symplectic homology, there are two "structure maps" which are invariant of the choice of topologically simple Liouville filling: in first structure map reads $\delta_\partial\colon SH^+_*(W) \to H_{n-*+1}(Y)$ (in our grading conventions), the second structure map is more complicated to state, but is defined in terms of the first. This provides an alternative proof of his result which does not rely on the connected sum argument (but still depends on Lazarev's result that ADC manifolds are invariant under surgery).

Looking ahead in a different direction, one would like to strengthen the results towards uniqueness of the filling. Some natural stepping stones towards this are the following.

Question 1. *Is every topologically simple Liouville filling of an ADC contact manifold of dimension at least five a Weinstein filling?*

For flexible Weinstein fillable manifolds, establishing this is a necessary part of proving uniqueness. Yet, two related results indicate that the answer might be negative. There are contact manifolds which are Liouville, but not Weinstein fillable [Bow12; BCS14]—which happens for topological reasons. Recently, Zhou found that there are also symplectic obstructions, since there exist manifolds which are Liouville fillable and almost Weinstein fillable (this is the necessary topological condition for a Weinstein filling), but not Weinstein fillable [Zho19]. It is not clear to the author whether and how these results apply to this question. In any case, the question can be bypassed by considering only Weinstein fillings.

If the previous question has an affirmative answer (or one restricts to Weinstein fillings), the next question is whether any filling must be flexible. We know a positive partial result: Cieliebak and Eliashberg showed that every Weinstein fillable contact manifold carries a flexibly fillable contact structure [CE12, Theorem 1.8; Laz18, Theorem 3.1].

Question 2. *If (M, ξ) is an ADC contact manifold of dimension at least five, is every Weinstein filling of (M, ξ) flexible?*

Vanishing of the symplectic homology is a necessary condition for flexibility, as Bourgeois, Eliashberg and Ekholm [BEE12; Ekh19] and also Murphy and Siegel [MS] showed. However, Murphy and Siegel showed that it is not sufficient: every flexible Weinstein domain W has a subdomain V_W which is not flexible. But $SH(W) = 0$ implies that $SH(V_W) = 0$ also (by Corollary 3.108).

Murphy and Siegel show that the subdomain V_W is not flexible by computing a modified version of symplectic homology (which they call *twisted symplectic homology*), showing that it doesn't vanish for V_W, but vanishes for all flexible domains. Yet, this only shifts the question: is a Weinstein domain flexible if and only if its symplectic homology, twisted symplectic homology (and other similar invariants) all vanish?

Finally, one arrives at a uniqueness question for flexible Weinstein fillings.

Question 3. *Is every flexible Weinstein filling of an asymptotically dynamically convex contact manifold of dimension at least five unique up to Weinstein homotopy?*

In terms of uniqueness, we see that applying a Liouville homotopy to a topologically simple filling still gives a topologically simple Liouville filling, so uniqueness of Liouville would hold up to Liouville homotopies. For Weinstein fillings, one could hope for a classification of fillings up to Weinstein homotopy.

In particular, Question 2 depends on the definition of flexible Weinstein domains: there is only a chance of an affirmative answer if flexible Weinstein domains are invariant under Weinstein homotopy—hence choosing the correct definition was important.

The recurring assumption of M being asymptotically dynamically convex of dimension at least five raises two follow-up questions.

Question 4. *Do these results generalise to contact manifolds which are not asymptotically dynamically convex?*

In greater generality, Lazarev's methods certainly do not apply any more, so this question seems more open to the author. Since there are contact manifolds with infinitely many Liouville fillings, there will be some limit for the results.

A. Appendix

A.1. Sign conventions used

An unfortunate fact in symplectic geometry is the proliferation of different incompatible sign conventions. Since several papers cited in this thesis use different conventions, we had to make a choice regarding them. We have chosen to mostly follow the sign conventions in Lazarev's and Zhou's papers [Laz17; Zho18] since these are the main works which we aim to explain—with the exception of the definition of the symplectisation. We have corrected the order of factors [Wen15], and also adopted the form $d(e^r\alpha)$ since that makes the definition of the completion more natural. (In turn, our calculations in Section 3.3 have to carry along a factor e^r. With sign conventions, there is no free lunch.)

object	**defining equation**
Hamiltonian vector field	$\omega(X_{H_t}, \cdot) = -dH_t$
Reeb vector field R_α	$d\alpha(R_\alpha, \cdot) = 0,\ \alpha(R_\alpha) = 1$
symplectisation	$(\mathbb{R} \times M, d(e^r\alpha))$
X_H and R_α on symplectisation	$X_H(r, p) = +h'(e^r)R_\alpha(p)$
symplectic action functional	$A_H(\gamma) = + \int_{\mathbb{S}^1} \gamma^*\omega - \int_{\mathbb{S}^1} H(t, \gamma(t))dt$
gradient flow	positive
Floer equation	$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0$
Conley-Zehnder and Morse index	$\mu_{CZ}(\gamma_p) = n - \mathrm{ind}(p)$

In these equations, (W, ω) is an exact $2n$-dimensional symplectic manifold and $(M, \xi = \ker\alpha)$ is a contact manifold. The symplectisation has $\mathbb{R}$-coordinate denoted by r and a Hamiltonian H which depends only on the $\mathbb{R}$-coordinate: $H(p, r) = h(r)$.

Signs which are unusual compared to personal taste or the literature are marked in red: we note that Lazarev and Zhou use the opposite sign for the symplectic action functional, in order to match the action of Reeb orbit (Definition 5.6) up to a constant term: since Reeb orbits shall have non-negative action, the first term must have a positive sign. To compensate (and still obtain the version of Floer's equation that one wants), one needs to consider the positive gradient flow instead, hence the action is *increasing* along Floer orbits.

In the literature, there is also a choice for the sign of the Conley-Zehnder index. The best way to make this visible is by comparing with the Morse index: the Conley-Zehnder index $\mu_{CZ}(\gamma_p)$ of a constant orbit γ_p at a critical point p is related to the Morse index $\operatorname{ind}(p)$ of p. With our conventions, we have $\mu_{CZ}(\gamma_p) = n - \operatorname{ind}(p)$. This gives precisely the grading relation in Theorem 3.32.

A.2. Different definitions of Weinstein domains

Unfortunately, there are several different definitions of Weinstein domains in the literature, and all definitions overlooked the same subtle issue. Fortunately, the issue was easy to correct (the definition we gave in Definition 2.75 was the correct one) and all definitions in the literature are equivalent. Since the equivalence is not obvious at first sight, we will explain it here.

First, we investigate the definitions we found in the literature. All definitions have in common that a Weinstein domain (W, ω, X, ϕ) is a Liouville domain, $\phi\colon W \to \mathbb{R}$ is a Morse function and X is gradient-like for ϕ. They all add one further assumption; we show that all of these assumptions are equivalent if ∂W is connected. More precisely, we have the following.

Proposition A.1. *Let (W, ω, X) be a Liouville domain, $\phi\colon W \to \mathbb{R}$ a Morse function and suppose X is gradient-like for ϕ. Suppose that ∂W is connected.*[1] *Then, the following are equivalent:*

- *(0) ϕ is constant on the boundary ∂W.*
- *(1) ∂W is a regular level set of ϕ: that is, we have $\partial W = \phi^{-1}(c)$ for some $c \in \mathbb{R}$ and c is a regular value of ϕ.*
- *(2) $\partial W = \phi^{-1}(c)$, where c is the maximal value of ϕ on W.*
- *(3) $\partial W = \phi^{-1}(c)$ is a regular level set of ϕ, and c is the maximal value of ϕ on W.*

We summarise these conditions as "∂W is a maximal regular level set of ϕ".

Among the papers we are dealing with, Lazarev [Laz17] and Zhou [Zho18] both assume that ∂W is "a maximal level set", presumably referring to the last assumption (2). In contrast, Cieliebak and Eliashberg [CE12] refer to assumption (1) in the introduction, and to assumption (0) as the actual definition.

For general Weinstein cobordisms or if ∂W is not connected, the same ideas will hold true, but the result needs natural adaptions. For example, every component of $\partial_\pm W$ will be contained in a regular level set. Let us now prove Proposition A.1.

[1] Morally, we mean to say "let (W, ω, X, ϕ) be Weinstein domain with ∂W connected", but the point of this result is exactly to investigate the final assumption in the definition of a Weinstein domain.

Proof. Clearly, all conditions imply condition (0). Since condition (3) is just the conjunction of conditions (1) and (2), we only need to prove the equivalence between (0) and (1) and (2). We show that (0) implies both (1) and (2).

The key argument is the following, which implies "(0) $\Rightarrow$ (2)".

Claim. *If ϕ is constant on the boundary ∂W, the value $c = \phi(\partial W)$ is the maximum value of ϕ on W, and we have $\phi^{-1}(c) = \partial W$.*

Proof of Claim. Since ϕ is continuous and W is compact, by Weierstraß' theorem ϕ assumes a maximum $\tilde{c}$ on W. Suppose there is an interior point p at which ϕ attains this maximum. Since ϕ is Morse, this implies that p is a critical point of ϕ, which has Morse index $2n$. However, since W is Weinstein, ϕ has no critical point of index $2n$, contradiction! Hence, we obtain $\phi^{-1}(\tilde{c}) \subset \partial W$. Moreover, the maximum value $\tilde{c}$ is attained on the boundary and we deduce $c = \tilde{c}$.[2] $\triangle$

We are left with proving that (0) implies (1). The claim shows that $\phi^{-1}(c) = \partial W$. If c were a critical value, the corresponding critical point had to lie in the interior: since X is transverse to the boundary, ϕ has no critical point on or near ∂W. However, the value c is attained only at the boundary, contradiction! Hence, c is a regular value and ∂W is a regular level set. $\square$

As we indicated after Definition 2.75, all definitions we found are slightly incorrect, since the boundary of a Weinstein domain need not be connected. However, this issue is subtle to overlook, since Weinstein domains in dimension at least four *do* have connected boundary:

Proposition A.2. *If (W, ω, X, ϕ) is a Weinstein domain of dimension $2n \geq 4$, the boundary ∂W is connected.*

Idea of proof. To prove this result, one uses the handlebody decomposition (see Proposition 2.125) of W, combined with the fact that all handles have index at most n: one can inductively construct W by starting with the ball and successively attaching handles of index at most n. At each step, if the boundary ∂W is connected, attaching a k-handle can only separate ∂W if $k \geq 2n - 1$; this can only happen for $n = 1$. $\square$

Remark A.3. There is a 2-dimensional Weinstein domain with disconnected boundary.

[2] In contrast to the interior, the place where $\tilde{c}$ is attained need not be a critical point.

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