ALGEBRA II, SHEET 9

Exercise 2

Let $L|K$ be a finite separable field extension. Then $S_L = \overline{K}^{\Xi}$ (the definition on the sheet is only relevant for the remark after the exercise). Now, both sides of the map (1) are $\overline{K}$-vector spaces of dimension $n = [L : K]$. Indeed, since $L|K$ is separable, $\#\Xi = n$. On the other hand, if $\alpha_1, \ldots, \alpha_n$ is a K-basis of L, then $\alpha_1 \otimes 1, \ldots, \alpha_n \otimes 1$ is a $\overline{K}$-basis of $R = L \otimes_K \overline{K}$. The map (1) maps this basis to $(\xi(\alpha_1))_{\xi\in\Xi}, \ldots, (\xi(\alpha_n))_{\xi\in\Xi}$. Thus, the $\overline{K}$-linear independence of Ξ tells us that (1) is injective, and hence also surjective. This is a fact from Galois theory, see `http://www.math.uconn.edu/~kconrad/blurbs/galoistheory/linearchar.pdf`.

Alternatively, write $L = K[X]/(f)$. Then $f = \prod_{i=1}^n (X - \alpha_i)$ in $\overline{K}[X]$, since it is separable. Therefore, by the Chinese remainder theorem,

$$R = K[X]/(f) \otimes_K \overline{K} = \overline{K}[X]/(f) = \prod_{i=1}^{n} \overline{K}[X]/(X - \alpha_i) \cong \overline{K}^{\Xi}.$$

Tracing through the maps, we can again see that (1) is an isomorphism.

If $L|K$ is algebraic and inseparable, $\operatorname{char} K = p$, then there exists some $\alpha \in L \setminus K$ such that $\alpha^p \in K$. Let $\beta \in \overline{K}$ be a root of $X^p - \alpha^p \in K[X]$. Then $x = \alpha \otimes 1 - 1 \otimes \beta \in R$ is a non-zero nilpotent element. Indeed, let $\varphi\colon L \to K$ be a K-vector space morphism with

$$\varphi(1) = 1 \text{ and } \varphi(\alpha) = 1$$

(which exists because $1, \alpha$ are K-linearly independent). Then $\varphi \otimes_K \overline{K}$ maps x to $1 - \beta \neq 0$. On the other hand,

$$x^p = \alpha^p \otimes 1 - 1 \otimes \beta^p = 1 \otimes (\alpha^p - \beta^p) = 0.$$

Again, we can also argue via the Chinese remainder theorem, choosing any inseparable $\alpha \in L$. Then the subring $K(\alpha) \otimes_K \overline{K} \subseteq R$ is of the form

$$K(\alpha) \otimes_K \overline{K} \cong \prod_{i=1}^{r} \overline{K}[X]/(X - \alpha_i)^{e_i}$$

with at least one $e_j \geq 2$. Thus $(0, \ldots, 0, X - \alpha_j, 0, \ldots, 0)$ is a non-zero nilpotent element.

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