A dichotomy for type A structures

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Introduction

Inspired by Zilber's trichotomy conjecture [16], a common theme in model theory has been to investigate, if non-linear behavior in model-theoretically tame structures leads to definability of fields. While Zilber's conjecture ultimately turned out to be false [12], other results in this direction turned out to be true. For instance in [10] Hieronymi and Walsberg consider the class of type A expansions of $(\mathbb{R}, <, +)$, constituting a vast generalization of o-minimality in this setting. An expansion of $(\mathbb{R}, <, +)$ is said to be *type A*, if it does not define a linearly ordered set $(X, \prec)$ of order type ω, such that $X \subseteq \mathbb{R}$ is dense in some open interval. Among other things they establish the following Zilber-style dichotomy.

Theorem ([10, Theorem A]). *Suppose $\mathcal{R}$ is a type A expansion of $(\mathbb{R}, <, +)$. Exactly one of the following statements holds:*

(a) $\mathcal{R}$ is field-type.

(b) Every D_Σ function $f\colon U \to R^n$, with $U \subseteq R^m$ open and definable, is locally affine on a dense open definable subset V of U.

Here $\mathcal{R}$ is said to be field-type, if there is a bounded open interval I and definable (which will always mean "definable with parameters") binary operations $\oplus, \otimes\colon I^2 \to I$ making $(I, <, \oplus, \otimes)$ isomorphic to the ordered field of real numbers $(\mathbb{R}, <, +, *)$. A D_Σ function is a function whose graph is a D_Σ set, which for now can be thought of as a definable analogue of an F_σ set.

The goal of this thesis is to extend the notion of type A from expansions of $(\mathbb{R}, <, +)$ to expansions of $(R, <, +)$, which in this thesis will always denote the additive ordered group of an ordered field $(R, <, +, *)$. We then derive the above and related results in this setting. We will now outline the obtained results.

We work in a structure $\mathcal{R}$ expanding $(R, <, +)$. We let $\mathcal{R}_*$ denote the structure we obtain by adding the multiplication $*$ of the ordered field $(R, <, +, *)$ to $\mathcal{R}$. We assume R to be equipped with the order topology. A set $X \subseteq R^n$ is said to be *pseudo-finite* if it is closed bounded and discrete. A linear order $(X, \prec)$, with $X \subseteq R^n$ is said to be a *pseudo ω-order* if every initial segment of the order is pseudo-finite. A structure $\mathcal{R}$ expanding $(R, <, +)$ is said to be of *type A*, if it does not define a pseudo ω-order $(X, \prec)$, with $X \subseteq R$ dense in some open interval and if additionally $\mathcal{R}_*$ is *definably complete*, that is every bounded $\mathcal{R}_*$-definable subset of

R has a supremum and an infimum. While it is not too hard to see that a pseudo ω-order is a first-order analogue of an ω-order in this context, it might not be immediately obvious, why one would only want to consider only those expansions $\mathcal{R}$, with $\mathcal{R}_*$ definably complete. When moving away from the real additive ordered group one loses topological completeness, so one should ask for some first-order analogue of topological completeness to be present and "definable completeness" certainly is such a first-order analogue, see e.g. [15] where the term is coined. But why is it not enough to only require $\mathcal{R}$ instead of $\mathcal{R}_*$ to be definably complete? We will return to this question after presenting the main results.

For the first result we have to introduce some notation. A set $A \subseteq R^n$ is D_Σ, if there is a definable family $(A_{s,t})_{s,t>0}$ of closed and bounded subsets of R^n, increasing in s and decreasing in t, such that $A = \bigcup_{s,t} A_{s,t}$. Using a definable version of the Baire Category Theorem for definably complete expansions of $(R, <, +, *)$, we are able to prove the following theorem, essentially by methods presented in [8] and [11].

Theorem A. *If $\mathcal{R}$ is type A, every D_Σ set either has interior or is nowhere dense.*

In Chapter 3 we extend a classical differentiability criterion from [1] due to Boas and Widder to a certain class of definably complete expansions of $(R, <, +*)$. Following [10] we use Theorem A and the differentiability criterion to prove:

Theorem 4.1. *Let $\mathcal{R}$ be a type A structure. Let $f\colon (a,b) \to R$ be a definable continuous function. Then f is C^k on a dense open subset of I.*

For a more succinct notation we will say that a property holds *generically* on an open definable set $U \subseteq R^n$ if the given property holds on a definable dense open subset V of U. For example the previous theorem says, that in type A structures, definable continuous functions $f\colon (a,b) \to R$ are generically C^k on (a,b). Generic C^k-smoothness of one variable functions will be the key tool to establish the Zilber-style dichotomy for the type A expansions of $(R, <, +)$.

Theorem B. *Let $\mathcal{R}$ be type A. Exactly one of the following statements holds:*

(a) $\mathcal{R}$ is field-type,

(b) Every D_Σ function $f\colon U \to R^n$, with $U \subseteq R^m$ open and definable, is generically locally affine on U.

As a corollary of this dichotomy we obtain that D_Σ functions in several variables are generically C^k.

Theorem C. *Let $\mathcal{R}$ be type A. Every D_Σ function $f\colon U \to R^n$, where $U \subseteq R^m$ is open, is generically C^k.*

Namely we first use that as a consequence of the SBCT, D_Σ functions are generically continuous to reduce the claim to definable continuous functions. The dichotomy now says that $\mathcal{R}$ is either not field-type, in which case all functions are generically C^∞, or $\mathcal{R}$ defines a field. Roughly said we then use that the field $\mathcal{R}$

defines, induces a differentiable structures on R^n, which is C^k-compatible with the usual differentiable structure induced by $(R, <, +, *)$ and an observation from 2.1 due to Fornasiero and Hieronymi to establish the theorem.

We now wish to return to the earlier question, why we want $\mathcal{R}_*$ to be definably complete, instead of just $\mathcal{R}$. For instance one could define a *type* A' structure to be a definably complete expansion of $(R, <, +)$ that does not define a dense pseudo ω-order, such that additionally $(R, <, +, *)$ is definably complete. Note that the condition of $(R, <, +, *)$ being definably complete is necessary, if one wants to have a chance to use the above version of being field-type, as the $\mathcal{R}$-definable field $(I, <, \oplus, \otimes)$ will inherit definable completeness from $\mathcal{R}$. A careful read of this thesis however will show, that all of the theorems listed in the introduction crucially use the fact that $\mathcal{R}_*$ is definable complete one way or another. So it seems quite unlikely that with this weaker type A notion the same results could be obtained by techniques similar to the presented ones. On the other hand the author could not come up with an example, proving that the proposed type A′ notion is to weak to yield Theorem B. We thus end this introduction with the following:

Question. Is there a type A′ structure, for which Theorem B does not hold?

Notation and Conventions

The natural numbers $\mathbb{N}$ are assumed to contain 0. If not declared otherwise k, m, n are assumed to be natural numbers.

If $f\colon X \to Y$ is a map of sets, we let $\Gamma_f = \{(x, y) \in X \times Y \colon f(x) = y\}$ be its graph.

$(R, <, +, *)$ will always be an ordered field and R will always denote its domain. We equip R with the order topology and powers of R are assumed to carry the product topology. For $x \in R$, $|x|$ denotes the absolute value of x, which is x, if $x \geq 0$ and $-x$ otherwise. For $x \in R^n$ we set $||x|| = \max\{|x_1|, \ldots, |x_n|\}$. Note that open balls, sets of the form $B_\epsilon(x) = \{y \in R^n \colon ||x - y|| < \epsilon\}$, for $x \in R^n$, $\epsilon \in R_{>0}$ form a base of the topology of R^n. If not declared otherwise ϵ and δ will be elements of R and I and J will be open subintervals of R.

$\mathcal{R}$ will always denote an expansion of $(R, <, +)$. Given a structure $\mathcal{R}$ we let $\mathcal{R}_*$ be the expansion of $\mathcal{R}$ one obtains by adding multiplication to $\mathcal{R}$. E.g. if $\mathcal{R} = (R, <, +)$, then $\mathcal{R}_* = (R, <, +, *)$. "Definable" always means "definable with parameters". To stress that some set is definable with respect to the structure $\mathcal{R}$ we will say that the set is $\mathcal{R}$-definable. Let $U \subseteq R^m$ be an open set.

If $A \subseteq R^{m+n}$ and $x \in R^m$ we put $A_x = \{y \in R^n \colon (x, y) \in A\}$. If $Z \subseteq R^m$ is a set, a family of $(A_z)_{z \in Z}$ of subsets of R^n is said to be definable, if the set $\{(x, z) \in R^{n+m} \colon z \in Z \text{ and } x \in A_z\}$ is definable. If $(A_z)_{z \in Z}$ is definable, the sets Z, $\bigcup_z A_z$, $\bigcap_z A_z$ and A_z for every $z \in Z$ are definable sets.

Chapter 1

Preliminaries

1.1 Definably complete structures

As a consequence of passing from expansions of $(\mathbb{R},<,+)$ to expansions of $(R,<,+)$, one loses topological completeness. A natural remedy for this problem is, to replace topological completeness by a suitable first order analogue, *definable completeness*. An expansion of $(R,<,+)$ is called *definably complete*, if every definable bounded subset of R has a supremum and an infimum. This concept was first introduced and investigated by Miller in [15]. It turns out that many results from elementary real analysis carry over to the definably complete setting, especially when considering definably complete expansions of ordered fields, as multiplication is crucial for the concept of differentiability. In this chapter we give a summary of relevant results. We first fix a definably complete expansion of $(R,<,+)$.

Definition. A set $A \subseteq R^n$ is called *definably connected*, if it is definable and whenever U and V are two definable subsets of A, open with respect to the subspace topology of A, with $A = U \cup V$, then $A = U$ or $A = V$.

Fact 1.1 ([15]). *Every interval, closed, open or half-open, is definably connected. If moreover $I_1, \dots, I_n$ are intervals (closed, open or half-open), the box $I_1 \times \dots \times I_n \subseteq R^n$ is definably connected.*

Fact 1.2 ([15]). *Every continuous definable function $f\colon [a,b] \to R$ has the Intermediate Value Property, that is it takes on all values between $f(a)$ and $f(b)$.*

Definition. A set $A \subseteq R^n$ is said to be *CBD*, if it is closed, definable and bounded.

If $R = \mathbb{R}$ the CBD sets are precisely the definable compact sets and in general a lot of desirable properties of compact sets translate to CBD sets. Among them, the following useful results.

Fact 1.3 ([15]). *(1) The image of a CBD set under a continuous definable map is CBD.*

(2) Let $(A_t)_{t>0}$ be a definable family of CBD sets which is either increasing or decreasing in t. Then $\bigcap_t A_t = \emptyset$ if and only if there is some $t > 0$ with $A_t = \emptyset$.

(3) *Every non-empty CBD set $A \subseteq R^n$ has a lexicographic minimum* $\operatorname{lexmin} A$, *which is an element of A inductively defined as follows. If $n = 1$, then* $\operatorname{lexmin} A = \min A$. *If $n > 1$ let $\pi\colon R^n \to R^{n-1}$ be the coordinate projection missing the last coordinate. Then* $\operatorname{lexmin} A = (x, \min A_x)$, *where* $x = \operatorname{lexmin} \pi(A)$.

We now turn towards differentiability. In order to do so, we fix a definably complete expansion of $(R, <, +, *)$. Many definitions carry over from elementary real analysis to the definably complete setting without modification. We will quickly review the relevant ones. The following material is taken from [3, Chapter 7]. Note that the reference assumes the structures to be o-minimal, but for the stated results, only definable completeness, if at all, is needed.

Let $f\colon I \to R^n$ be a function (not necessarily definable). It is said to be *differentiable in $x \in I$*, if there is $a \in R^n$, with

$$\lim_{t \to 0} \frac{f(x+t) - f(x)}{t} = a.$$

Note that then a is necessarily unique, so we may write $f'(x) = a$. Moreover f is necessarily continuous in x. If f is differentiable in all $x \in I$, it is said to be *a differentiable function*. The class of C^k functions on I is defined in the usual way. C^0 denotes the continuous functions and for $k > 0$ the class of C^k functions are those differentiable functions $I \to R^n$, whose derivative is a C^{k-1} function. If $f\colon I \to R^n$ is definable and differentiable, f' is definable as well, since we work in an expansion of $(R, <, +, *)$. If $U \subseteq R^m$ is open, f is a function $U \to R^n$, $x \in U$ and $1 \leq i \leq m$ the i-th partial derivative of f in x exists, if there is $a \in R^n$ with

$$\lim_{t \to 0} \frac{f(x+te_i) - f(x)}{t} = a,$$

where $e_i \in R^m$ denotes the i-th standard basis vector. Again, if the limit exists it is unique and we write $\partial_i f(x)$ for the *i-th partial derivative in x*. The class of C^k functions on U are defined similarly as in the one variable case. C^0 are the continuous functions on U and for $k > 0$, C^k will be the class of functions such that for all $1 \leq i \leq m$, $\partial_i f(x)$ exists for all $x \in U$ and $\partial_i f\colon U \to R^n$ is a C^{k-1} function.

Fact 1.4. *Let $f, g\colon I \to R$ be definable and differentiable in $x \in I$. Then $f + g$ and $f * g$ are differentiable in x and*

$$(f+g)'(x) = f'(x) + g'(x),$$
$$(f*g)'(x) = f'(x) * g(x) + f(x) * g'(x).$$

Fact 1.5 (Chain Rule). *Let $f\colon I \to R$, $g\colon J \to R^n$ be definable functions, such that f is differentiable in $x \in I$ and g is differentiable in $f(x) \in J$. Then $g \circ f$, which is defined on $f^{-1}(J)$ is differentiable in x and*

$$(g \circ f)'(x) = g'(f(x)) * f'(x).$$

As a corollary of the two preceding facts we obtain:

Fact 1.6. *Sums, products and compositions of definable C^k functions are definable C^k functions.*

Fact 1.7 (Mean Value Theorem)**.** *Let $f\colon [a,b] \to R$ be definable, continuous and differentiable on (a,b). Then there is $x \in (a,b)$ with $f(b) - f(a) = (b-a) * f'(x)$.*

Fact 1.8 (Theorem on Constants)**.** *Let $f\colon [a,b] \to R$ be definable, continuous and differentiable on (a,b). If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.*

Fact 1.9 (Inverse Function Theorem)**.** *Let $f\colon I \to J$ be C^k, $k \geq 1$ and bijective such that f' is nowhere zero, then $f^{-1}\colon J \to I$ is C^k.*

Proof. An application of the Mean Value Theorem shows that continuity of f' implies that f is strictly increasing or strictly decreasing. We may assume that f is strictly increasing, as the decreasing case can be proved by a minor change of the argument. We will follow the argument given in [14, Differentiaton der Umkehrfunktion, Chapter 9.2] to show that f^{-1} is differentiable in every $y_0 \in J$.

Let $x_0 \in I$ with $f(x_0) = y_0$. Since f is differentiable in x, there is a continuous definable function $\varphi\colon I \to R$ such that for all $x \in I$

$$f(x) - f(x_0) = \varphi(x) * (x - x_0)$$

and $f'(x_0) = \varphi(x_0)$. Since f is strictly increasing by assumption and its derivative in x_0 does not vanish, $\varphi(x) \neq 0$ for all $x \in I$. So for every $y \in I$ we have

$$f^{-1}(y) - f^{-1}(y_0) = \frac{1}{\varphi(f^{-1}(y))} * (y - y_0).$$

Continuity of $1/(\varphi \circ f^{-1})$ in y_0 implies that f^{-1} is differentiable in y_0 and its derivative is $1/f'(f^{-1}(y_0))$.

This moreover shows that f^{-1} is a C^1 function as its derivative is continuous as the composition of continuous functions. Using induction on $j \leq k$ and the fact, that the composition of C^j functions is C^j again, it follows that f^{-1} is C^j for every $j \leq k$, in particular f^{-1} is C^k. □

Fact 1.10. *Let $f\colon I \to R$ be differentiable and $f' \geq 0$. Then f is strictly increasing if and only if $\{x \in I\colon f'(x) = 0\}$ does not have interior.*

Proof. $f' \geq 0$ implies that f is increasing. Namely if f were decreasing, the Mean Value Theorem would provide an $x \in I$ with $f'(x) < 0$.

Suppose f is strictly increasing. Let $(a,b) \subseteq I$ be an interval. By the Mean Value Theorem, there is $x \in (a,b)$ with $f'(x) = \frac{f(b)-f(a)}{b-a} > 0$, so $(a,b) \not\subseteq \{x \in I\colon f'(x) = 0\}$, which implies that $\{x \in I\colon f'(x) = 0\}$ does not have interior.

Conversely suppose $\{x \in I\colon f'(x) = 0\}$ does not have interior and let $a < b \in I$. Then there is $x \in (a,b)$ where $f'(x) > 0$, so there is a small $\delta > 0$, such that $f(x+\delta) - f(x) > 0$. Hence $f(a) \leq f(x) < f(x+\delta) \leq f(b)$. □

1.2 D_Σ and definably meager sets

For this section we return to the setting of definably complete expansions of $(R, <, +)$.

Definition. A set $A \subseteq R^n$ is called a D_Σ *set* or simply D_Σ, if there is a definable family $(A_s, t)_{s,t>0}$ of CBD sets increasing in s and decreasing in t, such that $A = \bigcup_{s,t} A_{s,t}$. The family $(A_{s,t})_{s,t}$ is said to *witness* that A is D_Σ. A function is said to be D_Σ, if its graph is a D_Σ set.

Remark. Every D_Σ set is definable. Moreover it is easy to see that if $R = \mathbb{R}$, every D_Σ set is an F_σ set and in fact D_Σ sets should be viewed as a definable version of F_σ sets.

Fact 1.11. *(1) Open and closed definable sets are D_Σ.*

(2) Finite unions and finite intersections of D_Σ sets are D_Σ.

(3) If $A \subseteq R^{m+n}$ is D_Σ, then A_x is D_Σ for every $x \in R^m$.

(4) The image and the preimage of a D_Σ set under a continuous definable map is D_Σ. In particular a continuous definable function with a D_Σ domain, is D_Σ.

(5) If $A \subseteq R^{m+n}$ is D_Σ, then $\{x \in R^m : A_x$ has interior$\}$ is D_Σ.

Proof. For (1) to (4), see [2]. For (5), see [8, Fact 2.9 (1)]. While the proof in [8] is written for $R = \mathbb{R}$, the argument works without modification for arbitrary R. $\square$

Definition. A set $A \subseteq R^n$ is said to be *definably meager*, if there is a definable family $(A_{s,t})_{s,t>0}$ of nowhere dense sets, increasing in s and decreasing in t such that $X = \bigcup_{s,t} A_{s,t}$.

Remark. A D_Σ set is either definably meager or has interior.

The following two facts will be crucial ingredients for the proof of Theorem A in the next chapter.

Fact 1.12 ([7, Lemma 6.5]). *Let $A \subseteq R^{m+n}$ be a D_Σ set. Then A is definably meager if and only if $\{x \in R^m : A_x$ has interior $\}$ is definably meager. Equivalently A has interior if and only if $\{x \in R^m : A_x$ has interior $\}$ has interior.*

If $R = \mathbb{R}$, every definably meager set is meager in the ordinary sense, that is it is a countable union of nowhere dense sets. So when dealing with expansions of $(\mathbb{R}, <, +)$ the fact that $\mathbb{R}$ is a Baire space implies, that a definably meager set does not have interior. This result was extended by Hieronymi to definably complete expansions of ordered fields in [9].

Fact 1.13 (Baire Category). *Every definably meager set, definable in a definably complete expansion of an ordered field, does not have interior.*

Proof. See [4, Theorem 1.2] for a more general version of this theorem and a proof thereof. $\square$

Chapter 2

The Strong Baire Category Theorem

In this chapter we introduce type A expansions of $(R, <, +)$. They were first introduced by Hieronymi and Walsberg for expansions of $(\mathbb{R}, <, +)$ in [10]. Their definition extends naturally to arbitrary expansions of $(R, <, +)$. Following the work of Fornasiero and Hieronymi in [7], [8] and [11] we prove the Strong Baire Category Theorem (SBCT) for type A expansions of $(R, <, +)$.

Theorem A (SBCT). *If $\mathcal{R}$ is type A, every D_Σ set either has interior or is nowhere dense.*

Afterwards we collect some useful, straightforward corollaries thereof.

2.1 Type A structures

In [10] Hieronymi and Walsberg define *type A* expansions of $(\mathbb{R}, <, +)$ to be those structures that do not define a dense *ω-order*. Here a definable ω-order is a tuple $(X, \prec)$ consisting of a definable set $X \subseteq \mathbb{R}$ and a definable linear order $\prec$ on X, such that each initial segment of X with respect to the order $\prec$ is finite. An ω-order is said to be dense, if it is dense in some subinterval of $\mathbb{R}$. According to [10] "all the usual model-theoretic and geometric tameness notions in the literature imply type A" and the authors believe that type A structures constitute "the ultimate generalization of o-minimality in the setting of expansions of $(\mathbb{R}, <, +)$".

The fact that being finite is not first order definable in an arbitrary expansion of $(R, <, +)$ suggests that the naive generalization of type A to arbitrary expansions of $(R, <, +)$ is not the way to go. The property of being closed, bounded and discrete, which we will call *pseudo-finite*, is however definable in arbitrary expansions of $(R, <, +)$. If $R = \mathbb{R}$ the pseudo-finite sets are precisely the finite sets and moreover work by Fornasiero in [5] indicates that being closed, bounded and discrete is indeed a suitable first order analogue of finiteness in the context of definably complete expansions of $(R, <, +)$, justifying the name pseudo-finite. This motivates the following.

Definition. A set $X \subseteq R^n$ is called *pseudo-finite*, if it is closed bounded and discrete. A tuple $(X, \prec)$, with $X \subseteq R^n$ is called a *pseudo ω-order*, if $\prec$ is a linear order on X such that for every $x \in X$, $\{y \in X : y \prec x\}$ is a pseudo-finite set.

Definition. A structure $\mathcal{R}$ is said to be of *type A*, if $\mathcal{R}$ does not define a pseudo ω-order $(X, \prec)$, with $X \subseteq R$ dense in some subinterval of R and if additionally $\mathcal{R}_*$ is definably complete.

Remark. If $\mathcal{R}$ expands $(\mathbb{R}, <, +)$, the structure $\mathcal{R}_*$ is always definably complete, so $\mathcal{R}$ is type A in the sense of this thesis if and only if it is type A in the sense of [10].

In [7] Fornasiero and Hieronymi show that the definably complete expansions of an ordered field $(R, <, +, *)$ can be divided into two distinct classes, the *restrained* and the *unrestrained* expansions. A definably complete expansion of $(R, <, +, *)$ is called *unrestrained* if it defines a discrete subring $Z \subseteq R$, that is Z is a discrete definable subset of R and $(Z, +|_Z, *|_Z)$ is a subring of $(R, +, *)$. It is called *restrained* otherwise.

Definition. A type A structure $\mathcal{R}$ is called *restrained*, if its expansion $\mathcal{R}_*$ is restrained in the above sense. It is called *unrestrained* otherwise.

The dichotomy into restrained and unrestrained type A structures is a useful tool to prove statements about arbitrary type A structures. In order to show that a statement holds for every type A structure, we can now treat the restrained and unrestrained case separately. In the restrained case many of the statements this thesis is concerned with were already proved in [7], while in the unrestrained case the definability of Z, an analogue for the integers of $\mathbb{R}$, allows for the adaptation of proofs for type A expansions of $(\mathbb{R}, <, +)$ from [8], [10] and [11]. We will illustrate this technique with the proof of the SBCT, Theorem A. Namely the following result implies the SBCT for restrained type A structures.

Fact 2.1 ([7, Lemma 6.16])**.** *Let $\mathcal{R}$ be a definably complete restrained expansion of $(R, <, +, *)$. Every definably meager set is nowhere dense, in particular every D_Σ set is either nowhere dense or has interior.*

Thus we only need to prove the SBCT for unrestrained type A structures. Before we dive into the proof, we will collect and establish some elementary, useful results for unrestrained expansions of $(R, <, +, *)$, which will be needed in the proof of the SBCT and throughout this thesis.

For the remainder of this section we fix an unrestrained definably complete expansion of $(R, <, +, *)$. By [7], a definable discrete subring of R is necessarily unique and we denote its domain by Z. More precisely Z is the unique definable discrete subset of R, such that $(Z, +|_Z, *|_Z)$ is a nontrivial ring. Let N be the definable subset of Z consisting of the nonnegative elements and Q the definable field of fractions of Z.

Fact 2.2 ([7], Proposition 3.8)**.** *$(N, <, +, *)$ is a model of first-order Peano Arithmetic.*

We will use this to show that whenever we want to prove that some first-order property holds for all of N we can essentially do this by induction. We will call this style of proof *definable induction* or *Definable Induction Principle* to distinguish it from ordinary induction and to remind ourselves that we can only apply it to prove properties that are first order expressible in the language of the structure we are currently working in.

Lemma 2.3 (Definable Induction Principle)**.** *Let $A \subseteq R$ be a definable subset of N, that is inductive, so $0 \in A$ and whenever $n \in A$ also $n+1 \in A$. Then $A = N$.*

Proof. Using Fact 2.2 we see that $R \setminus N$ equals $\bigcup_{n \in N}(n, n+1) \cup (-\infty, 0)$, as for every $n \in N$ there is no element of N in between n and $n+1$. So N is a closed, discrete and definable subset of R. Towards a contradiction assume $N \setminus A$ was non-empty. Since $\mathcal{R}$ is definably complete and $N \setminus A$ is closed, definable and bounded below, $N \setminus A$ has a minimal element, say b. As $0 \in A$, b cannot be 0, which means that $b - 1 \in A$. But then $b \in A$, contradicting the fact that $b \in N \setminus A$. □

Fact 2.4. *There are definable surjective maps $N \to Z$, $N \to N^2$ and $N \to Q$.*

Proof. The definable map $Z \to N$, that sends z to $2z$ if $z \geq 0$ and to $-2z - 1$ otherwise is easily seen to be bijective using the Definable Induction Principle. Likewise the definable map $N^2 \to N$, $(x, y) \mapsto \frac{(x+y)(x+y+1)}{2} + x$ is easily seen to be bijective using the Definable Induction Principle. Finally the composition

$$N \to N^2 \to Z \times N \to Q,$$

where the map $Z \times N \to Q$ is given by $(z, n) \mapsto \frac{z}{n+1}$, is surjective and definable. □

Fact 2.5 ([7], Corollary 5.12)**.** *Let $c \in R$ and $g \colon R \to R$ be definable. There is a unique definable function $f \colon N \to R$, such that $f(0) = c$ and for all $n \in N$*

$$f(n+1) = g(f(n)).$$

Corollary 2.6. *Let $(c, i) \in R \times N$ and $g \colon R \times N \to R \times N$ definable. Then there exists a unique definable function $f \colon N \to R \times N$ such that $f(0) = (c, i)$ and for all $n \in N$*

$$f(n+1) = g(f(n)).$$

Proof. There is a definable bijection $\phi \colon (0, 1] \to (0, 1)$ given by $\phi(x) = x$, whenever $1/x \notin N$ and by $\phi(x) = 1/(1 + 1/x)$, whenever $1/x \in N$. This means there is a definable bijection $\psi \colon R \to R \times N$ given as the composition of the straightforward definable bijections

$$R \to (0, 1] \times Z \to (0, 1) \times Z \to (0, 1) \times N \to R \times N.$$

Let $g \colon R \times N \to R \times N$ be definable and $(c, i) \in R \times N$. Set $\tilde{g} = \psi^{-1} \circ g \circ \psi$. Using Fact 2.5 we get a definable map $\tilde{f} \colon N \to R$ with $\tilde{f}(0) = \psi^{-1}(c, i)$ and $\tilde{f}(n+1) = \tilde{g}(\tilde{f}(n))$ for all $n \in N$. It is easy to see that $f = \psi \circ \tilde{f}$ has the desired properties. Uniqueness of f follows using definable induction on n. □

Corollary 2.7. *Let $h\colon N \to R$ be a definable map. Then there is a unique definable map $H\colon N \to R$ such that $H(0) = h(0)$ and for all $n \in N$*

$$H(n+1) = H(n) + h(n+1).$$

Proof. Given h consider the definable map $g\colon R \times N \to R \times N$, sending (x, i) to $(x + h(i+1), i+1)$. By Corollary 2.6 there exists a unique definable map $f\colon N \to R \times N$ such that $f(0) = (h(0), 0)$ and for all $n \in N$

$$f(n+1) = g(f(n)).$$

Let $\pi\colon R \times N \to R$ be the projection onto the first coordinate and set $H = \pi \circ f$. Definable induction on n shows that $f(n) = (H(n), n)$, so H has the desired properties. □

If h and H are as described in Corollary 2.7, for every $n \in \mathbb{N} \subseteq N$ the equality $H(n) = \sum_{i=0}^{n} h(i)$ holds and in general H behaves just as we would expect a finite sum to behave (see Lemma 2.8). We therefore define:

Definition. Let $h\colon N \to R$ definable. Let H be as defined in Corollary 2.7. For every $n \in N$ we will write $\sum_{i=0}^{n} h(i)$ instead of $H(n)$.

Lemma 2.8. *Let $g, h\colon N \to R$ be definable functions and $c \in R$. For all $n \in N$ we have*

$$\sum_{i=0}^{n} g(i) + \sum_{i=0}^{n} h(i) = \sum_{i=0}^{n} g(i) + h(i),$$
$$\sum_{i=0}^{n} g(i) - \sum_{i=0}^{n} h(i) = \sum_{i=0}^{n} g(i) - h(i),$$
$$\sum_{i=0}^{n} c = (n+1) * c.$$

Proof. Definable induction on n. □

Definition. Let X be a set. An X *valued N-sequence* is a function $a\colon N \to X$ which we will denote by $(a_i)_{i \in N}$, where $a_i = a(i)$ for every $i \in N$.

Remark. By Fact 2.2 $(N, \leq)$ is a linear order, so in particular a directed set. This means that an N-sequence is a net. If X carries a topology we say an N-sequence *converges to* $a \in X$, if it converges to a as a net. Spelled out this equals the familiar notion of convergence of a sequence: $(a_i)_{i \in N}$ converges to $a \in X$, if for every neighbourhood U of a, there is $n \in N$ such that $a_i \in U$ for all $i \in N_{>n}$. If $f\colon X \to Y$ is a continuous map of topological spaces and $(a_i)_{i \in N}$ is an X valued N-sequence converging to $a \in X$, $(f(a_i))_{i \in N}$ is a Y valued N-sequence converging to $f(a)$. See [13, Chapters 2 and 3] for details.

Lemma 2.9. *For every $x \in R$ there is a Q valued N-sequence converging to x in R. In particular Q is dense in R.*

Proof. Using Fact 2.5 we can inductively define $1/2^n$ for every $n \in N$ by setting $1/2^0 = 1$ and if $1/2^n$ is already defined $1/2^{n+1}$ is defined to be $1/2 \cdot 1/2^n$. Using definable induction on n it is easy to see that $1/2^n \in Q_{>0}$ for all $n \in N$ and that for every $\delta > 0$, there is some $n \in N$ such that $1/2^i < \delta$ for all $i \in N_{>n}$.

To get an N-sequence in Q approximating some $x \in R$ it suffices to check that the definable set

$$A = \{n \in N\colon \text{ for all } x \in R \text{ there is } q \in Q \text{ with } q \leq x < q + 1/2^n\}$$

is inductive. $0 \in A$ since N is a closed, definable and discrete subset of R, so for every $x \in R$, the element $q = \sup N_{\leq x} \in N$ has the property $x \in [q, q+1)$, so $|x - q| < 1$. Now suppose that $n \in A$. Let $x \in R$ and choose $q \in Q$ with $q \leq x < q + 1/2^n$. Now either $q \leq x < q + 1/2^{n+1}$ or $q + 1/2^{n+1} \leq x < q + 1/2^n$. Since $1/2^{n+1} + 1/2^{n+1} = 1/2^n$ and $q + 1/2^{n+1} \in Q$, also $n + 1 \in A$. □

2.2 Proof of the Strong Baire Category Theorem

Recall that by Fact 2.1, Theorem A holds true whenever $\mathcal{R}$ is a restrained type A structure. So for the remainder of this section we assume that $\mathcal{R}$ is an unrestrained type A structure. Let N, Z and Q be the $\mathcal{R}_*$-definable subsets of R introduced in the last section.

Lemma 2.10 ([11, Lemma 3.1]). *Let $(X_t)_{t>0}$ be a definable family of pseudo-finite subsets of R, which is either increasing or decreasing in t. Then $\bigcup_t X_t$ is nowhere dense.*

Proof. As $\mathcal{R}$ is type A, it suffices to show that $X = \bigcup_t X_t$ admits a pseudo ω-order. We assume that $(X_t)_{t>0}$ is increasing as a slight modification of the argument proves the decreasing case. Let $\tau\colon X \to R_{>0}$ be the definable map, sending $x \in X$ to $\inf\{t > 0\colon x \in X_t\}$. For $x, y \in X$, we declare $x \prec y$ if $\tau(x) < \tau(y)$, or if $\tau(x) = \tau(y)$ and $x < y$. This defines a linear order on X and for every $x \in X$, the set $X_{\prec x}$ is pseudo-finite, as it is a subset of X_t for every $t > \tau(x)$. □

Definition. A subset $D \subseteq R_{>0}$ is called a *sequence set*, if it is bounded, discrete and its closure in R is $D \cup \{0\}$.

Remark. If $D \subseteq R_{>0}$ is a sequence set and $t \in D$, then $\{s \in D\colon s \geq t\}$ is pseudo-finite.

Lemma 2.11 ([11, Lemma 3.2]). *One of the following statements holds:*

(a) Every bounded nowhere dense definable subset of R is pseudo-finite.

(b) $\mathcal{R}$ defines a sequence set.

Proof. Suppose (a) does not hold. Then $\mathcal{R}$ defines a bounded nowhere dense set $X \subseteq R$ which is not discrete. As X is not discrete it has an accumulation point $x \in X$. Replacing X by $-x+X$ we may assume that the accumulation point is 0 and by replacing X by $-X$ if necessary, we may also assume that 0 is an accumulation point of $Y = X_{>0}$. We will now construct a definable sequence set D from Y. For every $\epsilon > 0$ let

$$L_\epsilon = \{\delta > 0 \colon \exists x, y \in Y_{\leq\epsilon} \text{ with } y = x + \delta \text{ and } Y \cap (x, y) = \emptyset\}$$

be the bounded definable set of lengths of complementary intervals of $Y_{\leq\epsilon}$. As Y is nowhere dense L_ϵ is nonempty for every $\epsilon > 0$. Let $l_\epsilon = \sup L_\epsilon$ and note that $R_{>0} \to R$, $\epsilon \mapsto l_\epsilon$ is a definable function. Let

$$M_\epsilon = \{x \in R \colon \exists\delta \in [\tfrac{l_\epsilon}{2}, l_\epsilon] \text{ with } x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2} \in Y_{\leq\epsilon} \text{ and } Y \cap (x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}) = \emptyset\}$$

be the definable nonempty set of midpoints of complementary intervals of $Y_{\leq\epsilon}$ of length between $l_\epsilon/2$ and l_ϵ. Let $x_\epsilon = \inf M_\epsilon = \min M_\epsilon$. Again $R_{>0} \to R$, $\epsilon \mapsto x_\epsilon$ is a definable function. Let $D = \{x_\epsilon \colon \epsilon > 0\}$. We will now check that D is a definable sequence set. It is easy to see, that D is a definable bounded subset of $R_{>0}$. Moreover $0 < x_\epsilon < \epsilon$, for every $\epsilon > 0$, so 0 is indeed an accumulation point of D. Finally D is discrete, as for every $\epsilon > 0$, x_ϵ is at least $\frac{l_\epsilon}{4}$ apart from every other element of D. □

Remark. Note that Lemma 2.11 precisely says that $\mathcal{R}$ does not define a sequence set if and only if every nowhere dense CBD subset of R is pseudo-finite.

Lemma 2.12 ([11, Lemma 3.3]). *Let $(X_t)_{t>0}$ be a definable family of nowhere dense subsets of R, either increasing or decreasing in t. Then $\bigcup_t X_t$ is nowhere dense.*

Proof. The proof given in [11] goes through word for word if one replaces each instance of finite by pseudo-finite and uses that by [6, Lemma 4.22] a pseudo-finite union of pseudo-finite sets is pseudo-finite again, as well as the following.

Claim. If $(C_t)_{t>0}$ is an $\mathcal{R}$-definable family of subsets of R either increasing or decreasing in t, such that $C = \bigcup_{t>0} C_t$ is dense in some bounded set A, then for every $\epsilon > 0$ there is $t > 0$ such that C_t is ϵ-dense in A, that is for every $a \in A$ there is $x \in C_t$ with $|x - a| < \epsilon$.

Proof of Claim. We only demonstrate the increasing case, as the decreasing case works analogously with very minor modifications. Let $\epsilon > 0$. Take $n \in N$ such that $\frac{1}{n} < \epsilon$. Let $B \subseteq Z$ be the $\mathcal{R}_*$-definable and bounded set of $z \in Z$ such that $(\frac{z-1}{n}, \frac{z+1}{n}) \cap A \neq \emptyset$. As C is dense in A for every $z \in B$ there is $t > 0$ such that $C_t \cap (\frac{z-1}{n}, \frac{z+1}{n}) \neq \emptyset$. Moreover, as the family $(C_t)_t$ is decreasing we have $C_s \cap (\frac{z-1}{n}, \frac{z+1}{n}) \neq \emptyset$ for every $0 < s < t$. Note that the map

$$f \colon B \to R_{>0} \cup \{\infty\},\ z \mapsto \frac{1}{2} \sup\{t \in R_{>0} \colon C_t \cap (\frac{z-1}{n}, \frac{z+1}{n}) \neq \emptyset\}$$

is $\mathcal{R}_*$-definable and that $C_{f(z)} \cap (\frac{z-1}{n}, \frac{z+1}{n}) \neq \emptyset$ for every $z \in B$. Using the Definable Induction Principle it is not hard to show, that $\inf_{z\in B} f(z) > 0$, as B is a pseudo-finite set. With $t = \inf_{z\in B} f(z)$, C_t is 2ϵ-dense in A. ■

□

Lemma 2.13 ([11, Lemma 3.4]). *Let $(X_{s,t})_{s,t>0}$ be a definable family of nowhere dense subset of R, increasing in s and decreasing in t. Then $\bigcup_{s,t} X_{s,t}$ is nowhere dense.*

Proof. This is a formal consequence of Lemma 2.12. See [11] for details. □

Proof of Theorem A. We will prove by induction on n, that every D_Σ set $A \subseteq R^n$ either has interior or is nowhere dense. To this end it suffices to prove that once A is somewhere dense, it has interior already. So suppose A is dense in the open set $U \subseteq R^n$. If $n = 1$, Lemma 2.13 implies that A has interior. Now suppose $n > 1$. After shrinking U if necessary we may suppose that $U = V \times I$, where V is a definable open set and I is some open interval. Since $A \cap U$ is a D_Σ set we may suppose that A is contained in U.

We will now show that

$$B = \{x \in V : A_x \text{ is dense in } I\}$$

is an $\mathcal{R}_*$-definably comeager subset of V. Fact 2.4 implies that there is a definable surjection $N \to \{(p,q) \in Q^2 : p < q\}$, $n \mapsto (l_n, u_n)$. For $n \in N$ define

$$C_n = \{x \in V : A_x \cap (l_n, u_n) \neq \emptyset\}$$

and note that $(C_n)_{n\in N}$ is an $\mathcal{R}_*$-definable family with $\bigcap_{n\in N} C_n = B$. Each C_n is D_Σ as it is the projection of the D_Σ set $A \cap V \times (l_n, u_n)$. As $A \cap V \times (l_n, u_n)$ is dense in $V \times (l_n, u_n)$, C_n is dense in V. As we have $\mathrm{cl}(V) = \mathrm{int}(C_n) \cup (\mathrm{cl}(C_n) \setminus \mathrm{int}(C_n))$ with both sets on the right hand side D_Σ the inductive assumption implies that $\mathrm{int}(C_n)$ is dense in V. For $n \in N$ set $D_n = V \setminus C_n$ and $\tilde{D}_n = \bigcup_{m\in N_{\leq n}} D_m$. Note that $(\tilde{D}_n)_{n\in N}$ is a definable family increasing in n. Definable induction on n shows that for each $n \in N$, $\tilde{D}_n$ is a nowhere dense set. We have

$$V \setminus B = \bigcup_{t>0} \tilde{D}_{\lfloor t \rfloor},$$

with $\lfloor t \rfloor = \sup\{n \in N : n \leq t\}$, so the set $V \setminus B$ is $\mathcal{R}_*$-definably meager. The Baire Category Theorem, Fact 1.13 implies that B cannot be $\mathcal{R}_*$-definably meager, as V is not definably meager.

For every $x \in B$, the inductive assumption implies that A_x has interior, so B is a subset of $\{x \in R^{n-1} : A_x \text{ has interior }\}$. Thus $\{x \in R^{n-1} : A_x \text{ has interior }\}$ is not definably meager and as it is D_Σ, it has interior. Fact 1.12 implies that A has interior. □

2.3 Corollaries of the strong Baire Category Theorem

The following results are consequences of the SBCT.

Proposition 2.14 ([8, Prop. 5.5])**.** *Suppose $\mathcal{R}$ is type A. Let $A \subseteq R^{m+n}$ be D_Σ such that $\pi(A)$ has interior, where $\pi\colon R^{m+n} \to R^m$ is the projection onto the first m coordinates. Then there is a definable open subset $V \subseteq R^m$ with $V \subseteq \pi(A)$ and a continuous definable $f\colon V \to R^n$ such that $\Gamma_f \subseteq A$.*

Theorem 2.15 ([10], Theorem 2.4)**.** *Suppose $\mathcal{R}$ is type A. Let $U \subseteq R^m$ be definable and open and let $f\colon U \to R^n$ be a D_Σ function. Then f is generically continuous on U.*

Theorem 2.16 ([10], Fact 2.5)**.** *Suppose $\mathcal{R}$ is type A. Let $Z \subseteq R^n$ be definable and let $(f_z\colon I_z \to R)_{z\in Z}$ be a definable family of continuous functions, where each I_z is an open interval. Then there is a definable family $(U_z)_{z\in Z}$ such that U_z is an open dense subsets of I_z and for every $z \in Z$ the function f_z is strictly increasing, strictly decreasing or constant on each definably connected component of U_z.*

We will start by proving Proposition 2.14.

Definition. Let $X \subseteq R^m$, $f\colon X \to R^n$ and $\epsilon > 0$. The function f is said to have *ϵ-oscillation* at $x \in X$ if for all $\delta > 0$ there are $y, y' \in X$ with $||x - y||, ||x - y'|| < \delta$ and $||f(y) - f(y')|| \geq \epsilon$.

Lemma 2.17 ([8], Lemma 5.2)**.** *Let $U \subseteq R^m$ be open and definable and $f\colon U \to R^n$ be definable. The set of points at which f is discontinuous is D_Σ. Furthermore one of the following holds:*

(a) There is a nonempty definable open $V \subseteq U$ such that $f|_V$ is continuous.

(b) There is a nonempty definable open $V \subseteq U$ and $\epsilon > 0$ such that f has ϵ-oscillation at every $x \in V$.

Proof. The proof given in [8] in case of $R = \mathbb{R}$ works for every type A structure. □

This is enough to prove Proposition 2.14. The proof goes through just as in [8] with the exception that we have to give a different argument to start the induction. For the convenience of the reader the whole proof is included.

Proof of Proposition 2.14. We will first reduce to the case that A is a CBD set. Let $(A_{s,t})_{s,t>0}$ be a definable family of subsets of R^{m+n} witnessing that A is D_Σ. Since projections of CBD sets are CBD again, $(\pi(A_{s,t}))_{s,t}$ witnesses that $\pi(A) \subseteq R^m$ is D_Σ. Since $\pi(A)$ has interior it is not definably meager, so there are $s, t > 0$ such that $\pi(A_{s,t})$ has interior. Replacing A by $A_{s,t}$ if necessary, we may assume that A is CBD. Take an open definable $U \subseteq \pi(A)$. Let $f\colon U \to R^n$ be the definable function that sends $x \in U$ to the lexicographic minimum of A_x. Note that $\Gamma_f \subseteq A$, by Fact 1.3.

We will now use induction on n to show that f is continuous on a nonempty open subset of U. Suppose $n = 1$. In this case $f(x) = \min A_x$ for all $x \in U$. Assume towards a contradiction that f is not continuous on some definable open subset $V \subseteq U$. Lemma 2.17 implies that there are a nonempty definable open $V \subseteq U$ and $\epsilon > 0$ such that f has 2ϵ-oscillation at every $x \in V$. The set $f(V) \subseteq R$ is bounded and definable so it has a supremum in R. Pick $x \in V$ such that $f(x)$ lies within $\epsilon/2$ of $\sup f(V)$. Let $X \subseteq V$ be a closed box with $x \in \mathrm{int}(X)$. Define $A' = A \cap X \times (-\infty, f(x) - \epsilon]$. Note that A' and hence $\pi(A')$ are CBD sets and that $x \notin \pi(A')$.

We will derive a contradiction by showing that x lies in the closure of $\pi(A')$. Since f has oscillation 2ϵ on V, for every $\delta > 0$ there is $x' \in X$, with $||x - x'|| < \delta$ and $|f(x) - f(x')| \geq \epsilon$. Since $f(x)$ lies within $\epsilon/2$ of $\sup f(V)$, $f(x') \leq f(x) - \epsilon$, so x' lies in $\pi(A')$. This shows that $x \in \mathrm{cl}(\pi(A'))$, a contradiction.

Suppose $n \geq 2$. Let $\rho\colon R^{m+n} \to R^{m+n-1}$ be the coordinate projection missing the last coordinate. Let $B = \rho(A)$. Note that B is a CBD set. Let $g\colon U \to R^{n-1}$ be the definable function sending x to the lexicographic minimum of $B_x \subseteq R^{n-1}$. By the definition of the lexicographic minimum $g(x) = \rho(f(x))$, for all $x \in U$. The inductive hypothesis implies, that g is continuous on some nonempty definable $U' \subseteq U$. By possibly shrinking U' we may assume that g is continuous on the closure of U'. Let $C \subseteq \mathrm{cl}(U') \times R$ be the set consisting of all (x, t) with $(x, g(x), t) \in A$. Note that C is CBD as it is a coordinate projection of the set CBD set $\Gamma_g \times R \cap A$. Let $h\colon U' \to R$, $x \mapsto \min C_x$. The same argument as in the case $n = 1$ shows, that h is continuous on some nonempty definable open $V \subseteq U$. Since $f(x) = (g(x), h(x))$ for every $x \in U'$, we obtain that $f|_V$ is continuous. □

Having established Proposition 2.14, we can derive Theorem 2.15 without much trouble.

Proof of Theorem 2.15. Let $D \subseteq U$ be the set of points where f is discontinuous. It suffices to show that D is nowhere dense, as this implies that $U \setminus \mathrm{cl}(D)$ is a dense open definable subset of U, restricted to which f is continuous. By Lemma 2.17, D is D_Σ, so the SBCT implies that D is either nowhere dense or has interior. Towards a contradiction assume hat D has interior. By replacing U with an open definable subset of D, we may assume that f is a D_Σ function which is nowhere continuous. Let $\pi\colon R^{m+n} \to R^m$ be the coordinate projection onto the first m coordinates. Since $\pi(\Gamma_f) = U$ has interior, there is a nonempty open definable V and a continuous definable $g\colon V \to R^n$ with $\Gamma_g \subseteq \Gamma_f$, by Proposition 2.14. This means that $f|_V = g$ is continuous, contradicting the assumption that f is nowhere continuous. □

To prove Theorem 2.16 we need a lemma, which is proved in [11] in the case of $R = \mathbb{R}$. Proposition 2.14 allows for a short proof for general R.

Lemma 2.18 ([11, Prop. 4.2])**.** *Suppose $\mathcal{R}$ is type A. Let $I \subseteq R$ be an open interval and $f\colon I \to R$ a nonconstant continuous definable function. Then there is an open subinterval of I on which f is strictly increasing or strictly decreasing.*

Proof. Continuous functions are D_Σ, so $A = \{(f(x), x)\colon x \in I\}$ is a D_Σ set. Let $\pi\colon R^2 \to R$ be the projection onto the first coordinate. The Intermediate Value Theorem implies that the projection $\pi(A) = f(I)$ has interior, so by Proposition 2.14 there is an open interval $J \subseteq f(I)$ and a continuous definable function $g\colon J \to R$ with $\Gamma_g \subseteq A$. This precisely means that, $f \circ g(y) = y$ for all $y \in J$. With $I' = f(J)$, we see that $g\colon J \to I'$ is a definable homeomorphism whose continuous inverse is $f\colon J \to I'$. This implies that I' is an open subinterval of I on which f is strictly increasing or strictly decreasing. $\square$

We can now follow the proof of [11, Thm. 4.3] to establish Theorem 2.16.

Proof of Theorem 2.16. Let $z \in Z$. Define the following disjoint subsets of I_z.

$$\begin{aligned}
U_{z,1} &= \{x \in I_z \colon f_z \text{ is strictly increasing on an open interval around } x\},\\
U_{z,2} &= \{x \in I_z \colon f_z \text{ is strictly decreasing on an open interval around } x\},\\
U_{z,3} &= \{x \in I_z \colon f_z \text{ is constant on an open interval around } x\}.
\end{aligned}$$

Set $U_z = U_{z,1} \cup U_{z,2} \cup U_{z,3}$. This is open and definable, as $U_{z,1}, U_{z,2}$ and $U_{z,3}$ are definable open sets. By Lemma 2.18 every open subinterval of I_z contains an open interval on which f_z is strictly increasing, strictly decreasing or constant. Thus U_z is a dense open subset of I_z. Since $\{U_{z,1}, U_{z,2}, U_{z,3}\}$ is a disjoint open cover of U_z every definably connected component of U_z is contained in $U_{z,i}$ for a unique $i \in \{1,2,3\}$ and is therefore a connected component of $U_{z,i}$. Because $(U_z)_{z\in Z}$ is a definable family it is left to show that for each $z \in Z$, f_z is strictly increasing on the definably connected components of $U_{z,1}$, strictly decreasing on the definably connected components of $U_{z,2}$ and constant on the definably connected components of $U_{z,3}$.

Let $C \subseteq U_{z,1}$ be a definably connected component of $U_{z,3}$. For every $x \in C$, consider the definable set $D_x = \{y \in C \colon y > x \text{ and } f_z(y) \leq f_z(x)\}$. If D_x were nonempty it had an infimum $y' \in C$. Since f_z is strictly increasing around x, it follows that $x < y'$. But f_z is also strictly increasing around y', so it cannot be the infimum of D_x, meaning that D_x is empty and f_z is in fact strictly increasing on C. The analogous argument shows that f_z is strictly decreasing on any definably connected component of $U_{z,2}$.

Finally let $C \subseteq U_{z,3}$ be a definably connected component of $U_{z,3}$ and let $x \in C$. Consider the definable sets $\{y \in C \colon f_z(y) = f_z(x)\}$ and $\{y \in C \colon f_z(y) \neq f_z(x)\}$, they are both open and definable and form a disjoint cover of C. Therefore the latter set is empty and f_z is indeed constant on C. $\square$

Chapter 3

A differentiability criterion

In this chapter we will establish Theorem 3.1, a differentiability criterion for a continuous function $f\colon (a,b) \to R$, definable in an unrestrained definably complete expansion $\mathcal{R}$ of a real closed field $(R,<,+,*)$. Importantly it is first order expressible in the structure $(R,<,+,f)$, so it will be the crucial ingredient in proving Theorem 4.1. For the field of real numbers, this differentiability criterion was proved by Boas and Widder in [1]. It turns out that essentially their technique carries over to unrestrained definably complete expansions of ordered fields. Throughout this chapter we fix a definably complete unrestrained expansion of $(R,<,+,*)$. We let N, Z and Q be the definable subsets of R described in Section 2.1.

3.1 The differentiablity criterion

To state the result we need two definitions first.

Definition. Let $f\colon (a,b) \to R$ be a function. For arbitrary $\delta \geq 0$ we define $\Delta_\delta^0 f = f$ and recursively for $k > 0$

$$\Delta_\delta^k f\colon (a, b-k\delta) \to R,\ x \mapsto \Delta_\delta^{k-1} f(x+\delta) - \Delta_\delta^{k-1} f(x).$$

To ease notation we will write $\Delta_\delta f$ instead of $\Delta_\delta^1 f$. Using induction on k we can establish the following formula needed later on in the proof:

$$\Delta_\delta^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+i\delta),\ \forall \delta \geq 0, \forall x \in (a, b-k\delta).$$

Definition. Let $a,b,c,d \in R$ with $a \leq c < d \leq b$. We say a function $f\colon (a,b) \to R$ satisfies H_k on (c,d), if $f|_{(c,d)}$ is continuous and $\Delta_\delta^k f(x) \geq 0$ for all $\delta > 0$ and $x \in (c, d-k\delta)$.

Theorem 3.1. *Let $\mathcal{R}$ be an unrestrained definably complete expansion of $(R,<,+,*)$ and let $k \geq 1$. Every definable function $(a,b) \to R$ that satisfies H_{k+2} on (a,b) is C^k.*

3.2 Proof of the differentiability criterion

We will now establish Theorem 3.1 by a series of lemmas, following the structure of the proof given in [1] closely. For the convenience of the reader we also include those proofs which carry over to our setting word for word. Until the end of this section f will always denote a definable function $(a, b) \to R$.

Lemma 3.2 ([1, Lemma 1]). *Let $k \geq 2$ and let f satisfy H_k on (a, b). For $\delta_1, \dots, \delta_k \geq 0$ and $x \in (a, b - \sum_{i=1}^{k} \delta_i)$ it holds*

$$\Delta_{\delta_1} \cdots \Delta_{\delta_k} f(x) \geq 0.$$

Proof. Let $n \in N_{>0}$, $h > 0$ and $x \in (a, b - h)$. Using definable induction on n we establish

$$\Delta_h f(x) = \sum_{i=0}^{n-1} \Delta_{h/n} f(x + ih/n).$$

Given k and $x \in (a, b - (k-1)h)$ the previous identity together with Lemma 2.8 and induction on k gives us

$$\Delta_h^{k-1} f(x) = \sum_{i_1=0}^{n-1} \cdots \sum_{i_{k-1}=0}^{n-1} \Delta_{h/n}^{k-1} f(x + (i_1 + \cdots + i_{k-1})h/n).$$

This together with Lemma 2.8 implies that for $x \in (a, b - h/n - (k-1)h)$ we have

$$\Delta_{h/n} \Delta_h^{k-1} f(x) = \sum_{i_1=0}^{n-1} \cdots \sum_{i_{k-1}=0}^{n-1} \Delta_{h/n}^{k} f(x + (i_1 + \cdots + i_{k-1})h/n) \geq 0. \tag{3.1}$$

If $m \in N$ and $x \in (a, b - mh/n - (k-1)h)$ inequality (3.1) and definable induction on m implies

$$\Delta_h^{k-1} f(x) \leq \Delta_h^{k-1} f(x + mh/n). \tag{3.2}$$

Given $\delta_1 > 0$ and $x \in (a, b - \delta_1(k-1)h)$ we can choose an N-sequence $(m_i/n_i)_{i \in N}$ with $m_i, n_i \in N_{>0}$ and $0 < m_i/n_i < \delta_1/h$, $\forall i \in N$ converging to δ_1/h. Continuity of f together with (3.2) implies

$$\Delta_h^{k-1} f(x) \leq \Delta_h^{k-1} f(x + \delta_1),$$

or

$$\Delta_h^{k-1} \Delta_{\delta_1} f(x) \geq 0. \tag{3.3}$$

If $k = 2$ this is the desired conclusion. Suppose $k > 2$ and that the lemma has been established for $k - 1$. Inequality (3.3) shows that for $\delta_1 \in (0, b - a)$, the function $x \mapsto \Delta_{\delta_1} f(x)$ satisfies H_{k-1} on $(a, b - \delta_1)$. The inductive hypothesis implies that for all $\delta_2, \dots, \delta_k \geq 0$ with $x \in (a, b - \sum_{i=0}^{k} \delta_i)$ we have

$$\Delta_{\delta_2} \cdots \Delta_{\delta_n} \Delta_{\delta_1} f(x) \geq 0.$$

Since $\delta_1 \in (0, b - a)$ can be chosen arbitrarily the conclusion of the lemma follows for k. $\square$

Lemma 3.3 ([1, Lemma 2]). *Let $k \geq 2$ and f satisfy H_k on (a, b). For arbitrary $\epsilon > 0$ the functions*

$$(a, b - (k-1)\epsilon) \to R, x \mapsto \Delta_\epsilon^{k-1} f(x),$$
$$(a + \epsilon, b - (k-2)\epsilon) \to R, x \mapsto \Delta_\epsilon^{k-1} f(x - \epsilon)$$

are non-decreasing.

Proof. Let $\epsilon > 0$ and let $y, z \in (a, b - (k-1)\epsilon)$ with $y < z$. Set $\delta = z - y$. Lemma 3.2 with $\delta_1 = \delta$ and $\delta_2 = \cdots \delta_n = \epsilon$ implies that

$$\Delta_\epsilon^{k-1} f(y) \leq \Delta_\epsilon^{k-1} f(z).$$

This shows that the first function is non-decreasing. This immediately implies that the second function is non-decreasing as it is merely a translated version of the first function. □

Lemma 3.4 ([1, Lemma 3]). *Let f satisfy H_2 on (a, b). For all $x \in (a, b)$,*

$$(0, b - x) \to R,\ h \mapsto \frac{f(x+h) - f(x)}{h}$$

is a non-decreasing function and

$$(0, x - a) \to R,\ h \mapsto \frac{f(x) - f(x-h)}{h}$$

is a non-increasing function.

Proof. We will only show that the first function is non-increasing as the argument for the second function is analogous. Let $x \in (a, b)$.

Claim. For all $m, n \in N$ and all $\epsilon > 0$ with $0 < m < n$ and $x + n\epsilon < b$ it holds

$$\frac{1}{m}\Delta_{m\epsilon} f(x) \leq \frac{1}{n}\Delta_{n\epsilon} f(x).$$

Proof of claim. Using definable induction this immediately reduce to the following claim: For all $m \in N_{>0}$ with $x + (m+1)\epsilon < b$ it holds

$$\frac{1}{m}\Delta_{m\epsilon} f(x) \leq \frac{1}{m+1}\Delta_{(m+1)\epsilon} f(x),$$

or equivalently

$$\Delta_{m\epsilon} f(x) \leq m\Delta_\epsilon f(x + m\epsilon). \tag{3.4}$$

We will now prove (3.4) by definable induction on m. The case $m = 1$ follows from Lemma 3.3. So let $m > 1$ and suppose (3.4) is already established for $m - 1$. Using this inductive hypothesis and Lemma 3.3 we obtain:

$$\begin{aligned}\Delta_{m\epsilon} f(x) &= \Delta_\epsilon f(x + (m-1)\epsilon) + \Delta_{(m-1)\epsilon} f(x)\\ &\leq \Delta_\epsilon f(x + m\epsilon) + (m-1)\Delta_\epsilon f(x + (m-1)\epsilon)\\ &\leq m\Delta_\epsilon f(x + m\epsilon).\end{aligned}$$

■

The claim applied to $\epsilon = \delta/n$ states that for all $m, n \in N$, $0 < m < n$ and all $\delta > 0$ with and $x + \delta < b$ it holds:

$$\frac{1}{m\delta/n}\Delta_{m\delta/n} f(x) \leq \frac{1}{\delta}\Delta_\delta f(x). \tag{3.5}$$

Given $0 < \epsilon < \delta$ and $x \in (a, b - \delta)$ choose an N-sequence $(m_i/n_i)_{i \in N}$, $m_i, n_i \in N_{>0}$, $m_i < n_i$ converging to ϵ/δ. Continuity of f together with (3.5) finally gives us

$$\frac{1}{\epsilon}\Delta_\epsilon f(x) \leq \frac{1}{\delta}\Delta_\delta f(x).$$

□

Lemma 3.5 ([1, Lemma 4]). *Let f satisfy H_2 on (a, b). $f'_+(x) = \lim_{\delta \to 0+} \frac{\Delta_\delta f(x)}{\delta}$ and $f'_-(x) = \lim_{\delta \to 0+} \frac{\Delta_\delta f(x-\delta)}{\delta}$ define non-decreasing functions $(a, b) \to R$ respectively.*

Proof. Lemma 3.4 implies that for every $x \in (a, b)$, $f'_+(x)$ and $f'_-(x)$ exist in $R \cup \{\pm\infty\}$, as $\mathcal{R}$ is definably complete. Let $\epsilon > \delta > 0$, $z \in (a + \epsilon, x)$ and $y \in (x, b - \epsilon)$. Using Lemma 3.3 and 3.4 we obtain

$$\frac{\Delta_\epsilon f(z-\epsilon)}{\epsilon} \leq \frac{\Delta_\epsilon f(x-\epsilon)}{\epsilon} \leq \frac{\Delta_\delta f(x-\delta)}{\delta} \leq \frac{\Delta_\delta f(x)}{\delta} \leq \frac{\Delta_\epsilon f(x)}{\epsilon} \leq \frac{\Delta_\epsilon f(y)}{\epsilon}.$$

This shows that $\frac{\Delta_\epsilon f(z-\epsilon)}{\epsilon} \leq f'_-(x) \leq f'_+(x) \leq \frac{\Delta_\epsilon f(y)}{\epsilon}$, so $f'_-(x), f'_+(x) \in R$ and $f'_-(z) \leq f'_-(x) \leq f'_+(x) \leq f'_+(y)$. In particular f'_- and f'_+ are non-decreasing. □

Lemma 3.6 ([1, Lemma 6]). *Let $c \in (a, b)$. If f satisfies H_2 on (c, b) and additionally $\lim_{x \to c+} f(x) = f(c)$ holds, $f'_+(c)$ exists in $R \cup \{\infty\}$. If f satisfies H_2 on (a, c) and additionally $\lim_{x \to c-} f(x) = f(c)$ holds, $f'_-(c)$ exists in $R \cup \{-\infty\}$.*

Proof. Say f satisfies H_2 on (c, b) and $\lim_{x \to c+} f(x) = f(c)$. Choose $\epsilon \in (0, b - c)$. By definable completeness of $\mathcal{R}$ it suffices to show that the definable function

$$d\colon (0, \epsilon) \to R,\ h \mapsto \frac{1}{h}\Delta_h f(c)$$

is non-decreasing. For large enough $n \in N$ the assignment $d_n(h) = \frac{1}{h}\Delta_h f(c + 1/n)$ gives a (definable) function $(0, \epsilon) \to R$, which is non-decreasing by Lemma 3.4. Now d is the pointwise limit of the N-sequence of functions d_n which are defined for large enough n. It is easy to check that d is non-decreasing.

An analogous argument works for the second case. □

The common hypothesis for Lemmas 3.7 through 3.10 is that $k \geq 3$ and f satisfies H_k on (a, b).

Lemma 3.7 ([1, Lemma 7]). *For any $x \in (a, b)$*

$$(0, \tfrac{b-x}{k-1}) \to R,\ h \mapsto \frac{1}{h^{k-1}}\Delta_h^{k-1} f(x)$$

is a non-decreasing function.

Proof. We prove the Lemma by induction on k. The base case $k = 2$ is Lemma 3.4. Now suppose $k > 2$ and the lemma has been established for $k - 1$. Let $\delta, h > 0$ and $x \in (a, b - 2\delta - (k-2)h)$. Lemma 3.2 with $\delta_1 = \delta_2 = \delta$, $\delta_3 = \cdots = \delta_k = h$ implies

$$\Delta_\delta^2 \Delta_h^{k-2} f(x) \geq 0,$$

so $\Delta_h^{k-2} f(x)$ satisfies H_2 on $(a, b-(k-2)h)$ and by Lemma 3.4

$$(0, b-(k-2)h-x) \to R,\ \delta \mapsto \frac{1}{\delta}\Delta_\delta \Delta_h^{k-2} f(x)$$

is a non-decreasing function. So for $x \in (a,b)$ and $\delta, \epsilon \in (0, \frac{b-x}{k-1})$, with $\delta < \epsilon$ we obtain

$$\frac{\Delta_\epsilon^{k-1} f(x)}{\epsilon^{k-1}} = \frac{1}{\epsilon^{k-2}} \frac{\Delta_\epsilon \Delta_\epsilon^{k-2} f(x)}{\epsilon} \geq \frac{1}{\epsilon^{k-2}} \frac{\Delta_\delta \Delta_\epsilon^{k-2} f(x)}{\delta} = \frac{1}{\delta} \frac{\Delta_\epsilon^{k-2} \Delta_\delta f(x)}{\epsilon^{k-2}} \tag{3.6}$$

Now let $\delta, h > 0$ and $x \in (a, b-(k-1)h-\delta)$. Lemma 3.2 with $\delta_1 = \cdots = \delta_{k-1} = h$, $\delta_k = \delta$ implies $\Delta_h^{k-1}\Delta_\delta f(x) \geq 0$, so

$$(a, b-\delta) \to R,\ x \mapsto \Delta_\delta f(x)$$

satisfies H_{k-1} on $(a, b-\delta)$. Using the induction hypothesis this implies that for $x \in (a,b)$ and $\delta, \epsilon \in (0, \frac{b-x}{k-1})$ with $\delta < \epsilon$ it holds

$$\frac{1}{\delta} \frac{\Delta_\epsilon^{k-2}\Delta_\delta f(x)}{\epsilon^{k-2}} \geq \frac{1}{\delta} \frac{\Delta_\delta^{k-2}\Delta_\delta f(x)}{\delta^{k-2}} = \frac{\Delta_\delta^{k-1} f(x)}{\delta^{k-1}}. \tag{3.7}$$

Combining (3.6) and (3.7) yields the conclusion of the Lemma for k. □

Lemma 3.8 ([1, Lemma 8]). *There is $c \in [a,b]$ such that f satisfies H_{k-1} on (c,b) and $-f$ satisfies H_{k-1} on (a,c).*

Proof. Consider the definable sets

$$\begin{aligned} A &= \{x \in (a,b) \colon \Delta_\delta^{k-1} f(x) \geq 0, \text{ for all } \delta \in (0, \tfrac{b-x}{k-1})\}, \\ B &= \{x \in (a,b) \colon \exists \delta > 0((k-1)\delta < b - x \wedge \Delta_\delta^{k-1} f(x) < 0)\}. \end{aligned}$$

A and B are disjoint and $A \cup B = (a,b)$. For every $y \in B$ there is $\delta > 0$, with $(k-1)\delta < b-y$ and $\Delta_\delta^{k-1} f(y) < 0$. By Lemma 3.3, $\Delta_\delta^{k-1} f(-)$ is non-decreasing so for every $z \in A$ necessarily $y < z$. Definable completeness of $\mathcal{R}$ ensures the existence of $c \in [a,b]$ with $(a,c) \subseteq B$ and $(c,b) \subseteq A$. By definition of A, f satisfies H_{k-1} on (c,b).

It remains to check that $-f$ satisfies H_{k-1} on (a,c). For every $x \in (a,c)$ there is $\epsilon > 0$ with $(k-1)\epsilon < b-x$ and $\Delta_\epsilon^{k-1} f(x) < 0$. Using Lemma 3.7 we see that $\Delta_\delta^{k-1} f(x) < 0$ for all $\delta \in (0,\epsilon)$ and using Lemma 3.3 we see that $\Delta_\epsilon^{k-1} f(y)$ for all $y \in (a,x)$. Consider the function

$$\epsilon\colon (a,c) \to R_{>0},\ x \mapsto \frac{1}{2}\sup\{\epsilon \in (0, \tfrac{b-x}{k-1}) \colon \Delta_\epsilon^{k-1} f(x) < 0\}.$$

Note that by the previous discussion ϵ is a non-increasing function and for all $x \in (a,c)$ it holds $\Delta^{k-1}_{\epsilon(x)} f(x) < 0$.

Let $\delta > 0$ and $x \in (a, c-(k-1)\delta)$. Choose $y \in (x+(k-1)\delta, c)$. Take some $n \in N$ with $\delta/n < \epsilon(y)$. The same argument as in the proof of Lemma 3.2 gives us the identity

$$\Delta^{k-1}_{\delta} f(x) = \sum_{i_1=0}^{n-1} \cdots \sum_{i_{k-1}=0}^{n-1} \Delta^{k}_{\delta/n} f(x + (i_1 + \cdots + i_{k-1})\delta/n).$$

Note that for every $i_1 + \cdots + i_{k-1}$ occurring in the above sum we have $\delta/n < \epsilon(y) \leq \epsilon(x + (i_1 + \cdots + i_{k-1})\delta/n)$ since ϵ is non-increasing and therefore every term in the above sum is smaller than 0. Using definable induction on n, one easily establishes $\Delta^{k-1}_{\delta} f(x) < 0$, so $-f$ satisfies H_{k-1} on (a,c). □

Lemma 3.9 ([1, Lemma 11]). *For all $x \in (a,b)$, $f'(x)$ exists in R.*

Proof. Using induction on k and Lemma 3.8 we find $a = x_0 < x_1 < \cdots < x_p = b$, $1 \leq p \leq 2^{k-1}$ such that for each $i \in \{0, \ldots, p-1\}$ either f or $-f$ satisfies H_2 on (x_i, x_{i+1}). This implies that for all $x \in \bigcup_{i=0}^{p-1} (x_i, x_{i+1})$, $f'_+(x)$ and $f'_-(x)$ exist in R by Lemma 3.5. For $i \in \{1, \ldots, p-1\}$, Lemma 3.6 implies that $f'_+(x_i)$ and $f'_-(x_i)$ exist in $R \cup \{\pm\infty\}$.

We will now show that $f'_+(x_i) \in R$ for $i \in \{1, \ldots, p-1\}$. Suppose $f'_+(x_i) = \infty$ for some i. For all $\delta > 0$ with $\delta < \frac{x_i - x_{i-1}}{k}$ and $\delta < x_{i+1} - x_i$ we have

$$\Delta^{k-1}_{\delta} f'_+(x_i - (k-2)\delta) = -\infty,$$

i.e.

$$\lim_{h \to 0+} \frac{1}{h} \Delta^{k-1}_{\delta} \Delta_h f(x_i - (k-2)\delta) = -\infty,$$

so for sufficiently small $h > 0$ we have $\Delta_h \Delta^{k-1}_{\delta} f(x_i - (k-2)\delta) < 0$, contradicting Lemma 3.2. If $f'_+(x_i) = -\infty$ for some i we let $0 < \delta < \frac{x_i - x_{i-1}}{k-1}$. Then

$$\Delta^{k-1}_{\delta} f'_+(x_i - (k-1)\delta) = -\infty,$$

an we reach a contradiction in similar fashion. This shows that $f'_+(x_i) \in R$ for all $i \in \{1, \ldots, p-1\}$

To conclude, we show that $f'_-(x) = f'_+(x)$ for all $x \in (a,b)$. So let $x \in (a,b)$ and let $h > 0$ with $h < \frac{x-a}{k}$, $h < \frac{b-x}{2}$. Let p be $k-2$ or $k-1$. Then

$$\Delta^k_h f(x - ph) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + (i-p)h) \geq 0,$$

and since we have $\sum_{i=0}^{k} (-1)^{k-i} = 0$ we obtain:

$$\sum_{\substack{i=0 \\ i \neq p}}^{k} (-1)^{k-i} \binom{k}{i} \frac{f(x + (i-p)h) - f(x)}{(i-p)h} (i-p) \geq 0.$$

Letting $h \to 0+$, we deduce

$$A_p f'_-(x) + B_p f'_+(x) \geq 0, \tag{3.8}$$

with

$$A_p = \sum_{i=0}^{p-1} (-1)^{k-i} \binom{k}{i} (i-p), \qquad B_p = \sum_{i=p+1}^{k} (-1)^{k-i} \binom{k}{i} (i-p).$$

A little calculation shows $A_p + B_p = 0$, so $A_p = -B_p$. If $p = k-2$, $A_p = k-2 > 0$, so $f'_-(x) \geq f'_+(x)$ and if $p = k-1$, $A_p = -1$, so $f'_+(x) \geq f'_-(x)$ by (3.8). In total $f'_+(x) = f'_-(x)$. □

Lemma 3.10 ([1, Lemma 13]). *$f' \colon (a,b) \to R$ satisfies H_{k-1} on (a,b).*

Proof. Let $h > 0$ and $x \in (a, b-(k-1)h)$. Then

$$\Delta_h^{k-1} f'(x) = \lim_{\delta \to 0+} \frac{\Delta_\delta \Delta_h^{k-1} f(x)}{\delta} \geq 0$$

by Lemma 3.2 so we are left showing continuity of f'.

Let $x \in (a,b)$. Using Lemma 3.8 k and induction on k, we find $\delta > 0$ such that f or $-f$ satisfies H_2 on $(x-\delta)$ and $(x, x+\delta)$. By Lemma 3.5 f' is monotonic on $(x-\delta)$ and $(x, x+\delta)$. Thus $f'(x+)$ and $f'(x-)$ exist in $R \cup \{\pm\infty\}$. With $h > 0$, $h < \frac{x-a}{k}$ and $h < \frac{b-x}{2}$ as well as $p \in \{k-2, k-1\}$ we have

$$\Delta_h^{k-1} f'(x - (p - \tfrac{1}{2})h) = \sum_{i=0}^{k-1} (-1)^{k-i-1} \binom{k-1}{i} f'(x + (i - p + \tfrac{1}{2})) \geq 0.$$

Letting $h \to 0+$ we obtain

$$A_p f'(x-) - A_p f'(x+) \geq 0,$$

where $A_p = \sum_{i=0}^{p-1} (-1)^{k-i-1} \binom{k-1}{i} = -\sum_{i=p}^{k-1} (-1)^{k-i-1} \binom{k-1}{i}$. We have $A_{k-2} = k - 2 > 0$ and $A_{k-1} = -1$ so we get:

$$f'(x+) = f'(x-).$$

Similarly using $\Delta_h^{k-1} f'(x - ph) \geq 0$ for sufficiently small $h > 0$ and again letting $h \to 0+$ gives us

$$B_p f'(x+) - B_p f'(x) \geq 0,$$

with

$$B_p = \sum_{\substack{i=0 \\ i \neq p}}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} = (-1)^{k-p} \binom{k-1}{p}.$$

Now $B_{k-2} = k-1$ and $B_p = -1$, so

$$f'(x) = f'(x+) = f'(x-),$$

which establishes continuity of f' on (a,b). □

Proof of Theorem 3.1. Induction on k using Lemma 3.10. □

Chapter 4

Differentiability of functions of one variable

Together with the tools we developed in the previous chapters we will now prove the key result towards the Zilber style dichotomy for type A structures.

Theorem 4.1. *Let $\mathcal{R}$ be a type A structure. Let $f\colon I \to R$ be a definable continuous function. Then f is C^k on a dense open subset of I.*

To prove Theorem 4.1 we again want to distinguish between restrained and unrestrained type A structures. In the restrained case the theorem immediately follows from the following result by Fornasiero and Hieronymi.

Fact 4.2 ([7, Lemma 6.19])**.** *Let $\mathcal{R}$ be a restrained definably complete expansion of an ordered field $(R, <, +, *)$. Let $U \subseteq R^n$ be open and definable and let $f\colon U \to R$ be a definable continuous function. Then f is C^k on a dense open definable subset of U.*

So we may assume that $\mathcal{R}$ is an unrestrained type A structure. Following section 4 of [10] closely, we will establish a stronger version of Theorem 4.1 for those structures. Namely we will show that a continuous definable function $f\colon I \to R$ is C^k on a dense open *definable* subset of its domain. The proof will require us to further divide the structures at hand into two distinct classes. Namely we will first prove Theorem 4.1 for those structures that do not define a sequence set. This will be done in Section 4.1. Afterwards we give a proof for those structures that define a sequence set in Section 4.2.

4.1 If $\mathcal{R}$ does not define a sequence set

As indicated above, for this section we assume $\mathcal{R}$ to be an unrestrained type A structure that does not define a sequence set. Note that by Lemma 2.11 every bounded nowhere dense definable subset of R is pseudo-finite. We begin by establishing some easy results towards the proof of Theorem 4.5.

Lemma 4.3. *Let $X \subseteq R$ be a pseudo-finite set. Then X is D_Σ and for every definable family $(A_{s,t})_{s,t>0}$ witnessing that X is D_Σ, there are $s, t > 0$ with $X = A_{s,t}$.*

Proof. For every $s, t > 0$ set $C_{s,t} = X \setminus A_{s,t}$. X being closed and discrete implies that each $C_{s,t}$ is CBD. The family $(A_{s,t})_{s,t>0}$ is increasing in s and decreasing in t, so $(C_{s,t})_{s,t>0}$ is decreasing in s and increasing in t. For every $s > 0$ set $D_s = \bigcap_t C_{s,t}$ and note that that $(D_s)_{s>0}$ is an increasing definable family of CBD sets. We have

$$\emptyset = \bigcap_{s,t} C_{s,t} = \bigcap_s \bigcap_t C_{s,t} = \bigcap_s D_s,$$

so by Fact 1.3 there has to be some $s > 0$ with $D_s = \emptyset$. Applying the same fact to the decreasing family $(C_{s,t})_{t>0}$ whose intersection is empty, we obtain $t > 0$, such that $C_{s,t} = \emptyset$. This precisely means that $X = A_{s,t}$. □

Lemma 4.4. *Let $X \subseteq I \times R_{>0}$ be D_Σ such that X_x is pseudo-finite for every $x \in I$. Then there is a nonempty open $J \subseteq I$ and $\epsilon > 0$ such that $J \times [0, \epsilon]$ is disjoint from X.*

Proof. Let $\pi \colon I \times R_{>0}$ be the projection onto the first coordinate. The projection $\pi(X)$ is still D_Σ so it is either nowhere dense or has interior by the SBCT. If it is nowhere dense, there is open nonempty $J \subseteq I$ disjoint from $\pi(X)$, so $J \times R_{\geq 0}$ is disjoint from X.

Now suppose that $\pi(X)$ has interior. By choosing an open subinterval $I' \subseteq \pi(X)$ and replacing X by the D_Σ set $X \cap I' \times R$, we may suppose that $\pi(X) = I$. Let $(A_{s,t})_{s,t>0}$ be a D_Σ family witnessing that X is D_Σ. For all $s, t > 0$ set

$$\begin{aligned} C_{s,t} &= \pi(X \setminus A_{s,t}), \\ D_{s,t} &= I \setminus C_{s,t}. \end{aligned}$$

Note that for every $x \in I$ we have $x \in D_{s,t}$ if and only if $X_x \subseteq (A_{s,t})_x$. For every $x \in I$, X_x is pseudo-finite and $((A_{s,t})_x)_{s,t>0}$ is a D_Σ family witnessing that it is D_Σ. Lemma 4.3 provides $s, t > 0$ with $X_x = (A_{s,t})_x$, hence $I = \bigcup_{s,t>0} D_{s,t}$. The SBCT implies that there are $s, t > 0$ such that $D_{s,t}$ is somewhere dense. Namely the definable family $(\mathrm{cl}(D_{s,t}))_{s,t>0}$ is a D_Σ family whose union has interior, so there have to be $s, t > 0$ such that $\mathrm{cl}(D_{s,t})$ has interior. Fix such s and t and an open nonempty $I' \subseteq I$ contained in the closure of $D_{s,t}$. It is easy to see that $C_{s,t}$ is D_Σ and since $D_{s,t} \cap I'$ is dense in I', $C_{s,t} \cap I'$ cannot have interior. Since $\mathcal{R}$ is SBCT type it is nowhere dense, implying that there is an open nonempty subinterval $J \subseteq I'$ whose closure is contained in $D_{s,t}$. Fix such a J.

Consider the definable continuous map $d \colon R^2 \times R^2 \to R_{\geq 0}, (x, y) \mapsto ||x - y||$. The CBD sets $\mathrm{cl}(J) \times \{0\}$ and $A_{s,t}$ are disjoint, so the CBD set $d(\mathrm{cl}(J) \times \{0\} \times A_{s,t})$ does not contain 0. Since $\mathcal{R}$ is definably complete there is $\epsilon > 0$ with $\epsilon < \inf d(\mathrm{cl}(J) \times \{0\} \times A_{s,t})$. This means $\mathrm{cl}(J) \times [0, \epsilon] \cap A_{s,t} = \emptyset$ and since we chose J in a way that

$$X \cap \mathrm{cl}(J) \times R = A_{s,t} \cap \mathrm{cl}(J) \times R,$$

we see that $J \times [0, \epsilon]$ is disjoint from X. □

Theorem 4.5. *Suppose $\mathcal{R}$ does not define a sequence set. Let $f\colon I \to R$ be a definable continuous function. Then there is a dense open definable subset U of I, such that f is C^k on U.*

Proof. Set

$$S = \{(x,h) \in I \times R_{>0} \colon x + (k+2)\delta \in I\},$$

and

$$W = \{(x,h) \in S \colon \Delta_h^{k+2} f(x) = 0\} \setminus \operatorname{int}\{(x,h) \in S \colon \Delta_h^{k+2} f(x) = 0\}.$$

Note that both sets are definable, S is open and W is closed in S, so W is D_Σ. Consider

$$Y = \{x \in I \colon W_x \text{ has interior}\},$$

which is D_Σ by Fact 1.11. W does not have interior, so Fact 1.12 implies that Y does not have interior. The SBCT implies that Y is nowhere dense, so $\tilde{U} = I \setminus \operatorname{cl}(Y)$ is open and dense in I.

Consider

$$V = \{x \in \tilde{U} \colon \forall \delta, \epsilon > 0 \text{ with } (x-\delta, x+\delta) \times (0,\epsilon) \cap W \neq \emptyset\}.$$

We will now show that V is nowhere dense. First note, that for all $x \in \tilde{U}$, W_x is pseudo-finite, as its closure is pseudo-finite by Lemma 2.11. Towards a contradiction suppose there is an open interval $J \subseteq I$, in which V is dense. The definable set $(J \times R_{>0}) \cap W$ is D_Σ so we can apply Lemma 4.4 to get an open subinterval $J' \subseteq J$ and $\epsilon > 0$ with $J' \times (0,\epsilon)$ disjoint from W. So V is disjoint from J' contradicting its density in J.

Finally set $U = I \setminus \operatorname{cl}(V)$, which is a dense open subset of I. It is left to show that f is C^k on U. Let $x \in U$. Since $x \notin V$ there are $\epsilon, \delta > 0$ such that $(x-\delta, x+\delta) \times (0,\epsilon) \cap W = \emptyset$. By decreasing ϵ and δ if necessary, we may assume that $(x-\delta, x+\delta) \times (0,\epsilon) \subseteq S$. The set $S \setminus W$ is the disjoint union of the three open definable sets $\{(x,h) \in S \colon \Delta_h^{k+2} f(x) > 0\}$, $\{(x,h) \in S \colon \Delta_h^{k+2} f(x) < 0\}$ and $\operatorname{int}\{(x,h) \in S \colon \Delta_h^{k+2} f(x) = 0\}$ and because $(x-\delta, x+\delta) \times (0,\epsilon)$ is definably connected it has to be contained in exactly one of the three open sets. If necessary decrease δ so that $2\delta < (k+2)\epsilon$. Now it is easy to check, that f satisfies H_{k+2} on $(x-\delta, x+\delta)$, so Theorem 3.1 implies that f is C^k around x. $\square$

4.2 If $\mathcal{R}$ defines a sequence set

It remains to treat the case that $\mathcal{R}$ is an unrestrained type A structure that defines a sequence set. First we establish a couple of auxiliary results.

Lemma 4.6. *Let D be a definable sequence set and let $\{X_d \colon d \in D\}$ be a definable family of subsets of R^n, such that X_d is nowhere dense for each $d \in D$. Then $\bigcup_{d \in D} X_d$ is nowhere dense.*

Proof. For every $t \in R_{>0}$ define $Y_t = \bigcup_{c \in D_{\geq t}} X_c$. Since $(X_c)_{c \in D_{\geq t}}$ is a pseudo-finite family of nowhere dense sets, being nowhere dense is a definable property and finite unions of nowhere dense sets are nowhere dense [6, Lemma 4.22] implies that each Y_t is nowhere dense. The SBCT implies that $\bigcup_{t>0} Y_t$ is nowhere dense. $\square$

For the proof of Theorem 4.9 we need a refined version of k-th difference $\Delta^k_\delta f$ of a function $f\colon I \to R$ which we defined in Chapter 3.

Definition. Let $f\colon (a,b) \to R$. Let $k > 0$ and let $h \in R^k_{\geq 0}$. Note that $\Delta^1_h f$ has already been defined in Chapter 3. For $k > 1$ define

$$\Delta^k_h f\colon (a, b - \sum_{i=1}^{k} h_i) \to R,\ x \mapsto \Delta^{k-1}_{(h_1,\ldots,h_{k-1})} f(x + h_k) - \Delta^{k-1}_{(h_1,\ldots,h_{k-1})} f(x).$$

Lemma 4.7. *Let $k > 0$, let $f, g\colon I \to R$ be definable, let $h = (u,v) \in R_{\geq 0} \times R^{k-1}_{\geq 0}$ and let $x \in I$ with $x + u + \sum_i v_i \in I$. Then:*

(1) $\Delta^k_{(u,v)} f(x) = \Delta^{k-1}_v \Delta_u f(x)$.

(2) $\Delta^k_h (f+g)(x) = \Delta^k_h f(x) + \Delta^k_h g(x)$.

(3) $\Delta^k_h (-f)(x) = -\Delta^k_h f(x)$.

Proof. Induction on k. $\square$

Lemma 4.8. *Let $f\colon I \to R$ be a continuous definable function and let D be a definable sequence set. If $\Delta^k_{(d,h)} f(x) \geq 0$ for all $d \in D$, $h \in R^{k-1}_{\geq 0}$ and $x \in I$ with $x + d + \sum_i h_i$, then $\Delta^k_w f(x)$ for all $w \in R^k_{\geq 0}$ and $x \in I$ with $x + \sum_i w_i \in I$, that is f satisfies H_k on I.*

Proof. Let $S_{I,k} = \{(x,w) \in I \times R^k_{\geq 0} \colon x + \sum_i w_i \in I\}$. To establish the desired result it suffices to prove that the closed set $\{(x,w) \in S_{I,k} \colon \Delta^k_w f(x) \geq 0\}$ is dense in $S_{I,k}$.

To prove density we want to use the definable induction principle discussed in section 2.1. Therefore we pass to the unrestrained structure $\mathcal{R}_*$ and denote the non-negative elements of the unique discrete definable subring by N just as in Section 2.1. Take an open, nonempty $U \subseteq S_{I,k}$ and let $(u,v,x) \in U$, where $u \in R_{\geq 0}$ and $v \in R^{k-1}_{\geq 0}$. Since U is open, there is $\delta > 0$ with $(u - \delta, u + \delta) \times \{v\} \times \{x\} \subseteq U$. Pick a $d \in D$ with $d < \delta$. Note that $\{m \in N \colon md \leq u\}$ is bounded above and definable in $\mathcal{R}_*$, so by definable completeness of $\mathcal{R}_*$ it has a supremum n. From $nd \leq u$ and $u < (n+1)d$ it follows that $0 \leq u - nd < d$, so $(nd, u, x) \in U$.

Therefore it suffices to prove that for all $m \in N$ with $1 \leq m \leq n$ we have $(md, v, x) \in S_{I,k}$ and $\Delta^k_{(nd,v)} f(x) \geq 0$. The first result follows from the fact that for all $m \in N_{\leq n}$

$$I \ni x < x + md + \sum_i v_i < x + nd + \sum_i v_i \in I.$$

We will establish the second result by definable induction on m. The case $m = 1$ follows from the assumption. Now suppose $m < n$. Using the inductive hypothesis, Lemma 4.7 and the fact that by assumption $\Delta^k_{(d,v)} f(x + md) \geq 0$, we obtain

$$\begin{aligned}
\Delta^k_{((m+1)d,v)} f(x) &= \Delta^{k-1}_v \Delta_{(m+1)d} f(x) \\
&= \Delta^{k-1}_v (f(x + md + d - f(x)) \\
&= \Delta^{k-1}_v (\Delta_d f(x + md) + \Delta_{md} f(x)) \\
&= \Delta^{k-1}_v \Delta_d f(x + md) + \Delta^{k-1}_v \Delta_{md} f(x) \\
&= \Delta^k_{(d,v)} f(x + md) + \Delta^k_{(md,v)} f(x) \geq 0.
\end{aligned}$$

The definable induction principle establishes the second claim and hence the lemma. □

Now we can give a proof of Theorem 4.1 for the structures at hand. We will actually prove the following uniform version of the theorem, which is tailored to the style of proof.

Theorem 4.9. *Suppose $\mathcal{R}$ defines a sequence set. Let $Z \subseteq R^n$ be definable, let $(I_z)_{z \in Z}$ be a definable family of bounded open intervals and let $(f_z \colon I_z \to R)_{z \in Z}$ be a definable family of continuous functions. Then there is a definable family $(U_z)_{z \in Z}$ of open dense subsets of I_z such that f_z is C^k on U_z for each $z \in Z$.*

Proof. For every $z \in Z$ write $I_z = (a_z, b_z)$. By Theorem 3.1 it suffices to show that for every $k \geq 1$ and every definable family $(f_z \colon I_z \to R)_{z \in Z}$ there is a definable family $(U_z)_{z \in Z}$, with $U_z \subseteq I_z$ open and dense, such that for every $z \in Z$ and every definably connected component J of U_z we have

(a) $\Delta^k_h f_z(x) \geq 0$ for all $h \in R^k_{\geq 0}$ and $x \in J$ with $x + \sum_i h_i \in J$, or

(b) $\Delta^k_h f_z(x) \leq 0$ for all $h \in R^k_{\geq 0}$ and $x \in J$ with $x + \sum_i h_i \in J$.

We will achieve this by induction on k. The Weak Monotonicity Theorem, Theorem 2.16 provides a family $(U_z)_{z \in Z}$ which has the desired properties for $k = 1$.

Now let $k > 1$ and let D be a definable sequence set. Applying the inductive hypothesis to the definable family

$$(\Delta_d f_z \colon (a_z, b_z - d) \to R,\, x \mapsto f_z(d + x) - d_z(x))_{(z,d) \in Z \times D}$$

we obtain a definable family $(U_{z,d})_{(z,d)}$, where $U_{z,d} \subseteq (a_z, b_z - d)$ is dense and open and for each connected component J of $U_{z,d}$ we have

(a) $\Delta^{k-1}_h \Delta_d f_z(x) \geq 0$ for all $h \in R^{k-1}_{\geq 0}$ and $x \in J$ with $x + \sum_i h_i \in J$, or

(b) $\Delta^{k-1}_h \Delta_d f_z(x) \leq 0$ for all $h \in R^{k-1}_{\geq 0}$ and $x \in J$ with $x + \sum_i h_i \in J$.

For each $(z,d) \in Z \times D$ set

$$X_{z,d} = \{b_z - d\} \cup (a_z, b_z - d) \setminus U_{z,d}.$$

Each $X_{z,d}$ is nowhere dense, so by Lemma 4.6 $\bigcup_{d \in D} X_{z,d}$ is nowhere dense for each $z \in Z$. Set

$$U_z = I_z \setminus \mathrm{cl}(\bigcup_{d \in D} X_{z,d}).$$

Note that $(U_z)_{z \in Z}$ is a definable family of open sets, each U_z being dense in I_z.

We will now show that for each $z \in Z$ and each connected component J of U_z, f satisfies H_k on J. To this end let $J \subseteq U_z$ be such a connected component. For each $d \in D$ we have $b_z - d \notin U_z$, so either $J \subseteq (b_z - d, b_z)$ or $J \subseteq (a_z, b_z - d)$. Let

$$D' = \{d \in D \colon J \subseteq (a_z, b_z - d)\}.$$

Since J has a positive length δ, $D_{\leq \delta} \subseteq D'$, which implies that D' is a sequence set. Observe that for every $d \in D'$ we have $J \subseteq U_{z,d}$. This means that together with the two sets

$$D'_{\geq} = \{d \in D' \colon \Delta_h^{k-1} \Delta_d f_z(x) \geq 0 \text{ for all } (h,x) \in R_{\geq 0}^{k-1} \times J \text{ with } x + \sum_i h_i \in J\},$$
$$D'_{\leq} = \{d \in D' \colon \Delta_h^{k-1} \Delta_d f_z(x) \leq 0 \text{ for all } (h,x) \in R_{\geq 0}^{k-1} \times J \text{ with } x + \sum_i h_i \in J\}$$

we have $D' = D'_{\geq} \cup D'_{\leq}$ implying that $D'_{\geq}$ or $D'_{\leq}$ has to be a sequence set.

Suppose $D'_{\geq}$ is a sequence set. We want to show, that $\Delta_w^k f(x) \geq 0$ for all $(w,x) \in R_{\geq 0}^k \times J$ with $x + \sum_i w_i \in J$. By Lemma 4.8 it suffices to show that $\Delta_{(d,h)}^k f(x) \geq 0$ for all $(d,h,x) \in D'_{\geq} \times R_{\geq 0}^{k-1} \times J$ with $x + d + \sum_i h_i \in J$. If (d,h,x) is such a triple, then $x + \sum_i h_i \in J$, so by definition of $D'_{\geq}$ together with Lemma 4.7 we obtain

$$\Delta_{(d,h)}^k f_z(x) = \Delta_h^{k-1} \Delta_d f_z(x) \geq 0.$$

If $D'_{\leq}$ is a sequence set the same argument works to show that $\Delta_w^k(-f(x)) \geq 0$ for all $(w,x) \in R_{\geq 0}^k \times J$ with $x + \sum_i w_i \in J$, so by Lemma 4.7, $\Delta_w^k f(x) \leq 0$ for all $(w,x) \in R_{\geq 0}^k \times J$ with $x + \sum_i w_i \in J$. $\square$

Chapter 5

The dichotomy

With the aid of Theorem 4.1, we will now establish the dichotomy for type A structures, Theorem B. To state the result we need the following definitions.

Definition. An expansion of $(R, <, +)$ is called *field-type* if there is a bounded open interval I and definable functions $\oplus, \otimes \colon I^2 \to I$ such that $(I, <, \oplus, \otimes)$ is an ordered field isomorphic to $(R, <, +, *)$.

Definition. Let $U \subseteq R^m$ be open and $f \colon U \to R^n$. The function f is said to be *locally affine* if every $x \in U$ has a neighbourhood, restricted to which f is of the form $x \mapsto Ax + b$, where A is some R-valued $n \times m$ matrix and $b \in R^n$.

Theorem B. *Let $\mathcal{R}$ be type A. Exactly one of the following statements holds:*

(a) $\mathcal{R}$ is field-type,

(b) Every D_Σ function $f \colon U \to R^n$, with $U \subseteq R^m$ open and definable, is generically locally affine on U.

The proof is split into two parts. We will first find a better characterization of the structures that exhibit linear only behavior in Proposition 5.1 and use this characterization to determine the field-type structures in Theorem 5.5. Throughout this chapter we assume that $\mathcal{R}$ is a type A structure.

5.1 Affine expansions

We start by giving a different characterization for linearity of type A structures.

Proposition 5.1. *Let $\mathcal{R}$ be type A. The following statements are equivalent:*

(i) Every continuous definable function $I \to R^n$ is generically locally affine on I.

(ii) Every D_Σ function $f \colon U \to R^n$, with $U \subseteq R^m$ open and definable, is generically locally affine on U.

We collect a few easy result needed to establish the proposition.

Lemma 5.2. *Let $U \subseteq R^m$ be open and definable. Any definable function $f\colon U \to R^n$, which is locally affine on some dense subset of U, is generically locally affine.*

Proof. Let $V \subseteq U$ be the set of points in which f is locally affine. Note that V is open. We will show that V is definable. For $x \in R^m$ and $\epsilon > 0$ let $B_\epsilon(x)$ be the open box $(x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_m - \epsilon, x_m + \epsilon)$. Let V' be the set of $x \in U$ with the following property. There is $\epsilon > 0$ with $B_\epsilon(x) \subseteq U$ such that for every $y, y' \in B_\epsilon(x)$ there is $\delta > 0$ with $B_\delta(y) \subseteq U$ and $B_\delta(y') \subseteq U$ such that for every $d \in B_\delta(0)$ the equality $f(y+d) - f(y) = f(y'+d) - f(y')$ holds.

The set V' is definable and open and contains all points from V. If conversely $x \in V'$, the density of V in U implies that f is locally affine at x, so $V = V'$. □

Lemma 5.3. *Let $f\colon I \to R$ be a definable continuous function, that is locally affine on I. Then f is affine on I.*

Proof. We may pass to the definably complete structure $\mathcal{R}_*$, where the slope of f is definable at every $x \in I$. Since the slope is locally constant, it has to be constant on all of I, as every interval is definably connected. Say $a \in R$ is the slope of f. Then $g\colon I \to R$, $x \mapsto f(x) - ax$ is a locally constant definable function. As I is definably connected, g is constant, proving that f is affine on I. □

The following lemma is exactly [10, Lemma 5.3]. Since the proof is written for $R = \mathbb{R}$, we recall it here to demonstrate that it works for arbitrary type A structures.

Lemma 5.4. *Let $A \subseteq R^m \times R$ be definable such that $A_t \subseteq R^m$ is locally closed and bounded for all $t \in R$. Let π be the coordinate projection $R^m \times R \to R$ onto the last coordinate. Then there is a definable function $g\colon \pi(A) \to R^m$ such that $(g(t), t) \in A$ for all $t \in \pi(A)$.*

Proof. For each $t \in \pi(A)$ let W_r be the union of open boxes $B \subseteq R^m$ of edge length at most 1 such that $B \cap A_t$ is nonempty and closed in B. The definable family $(W_t)_{t \in \pi(A)}$ consits of bounded open sets such that $A_t \subseteq W_t$ and A_t is closed in W_t for every $t \in \pi(A)$. Let

$$d\colon R^m \times R^m \to R, (x, y) \mapsto ||x - y||$$

which is a continuous definable function and observe that $x \mapsto d(x, R^m \setminus W_t) = \inf\{d(x, y)\colon y \in R^m \setminus W_t\}$ describes a continuous definable function $R^m \to R_{\geq 0}$. Moreover the infimum is actually a minimum. Now let $C \subseteq R^m \times R$ be the definable set of $(x, t) \in A$ such that

$$d(x, R^m \setminus W_t) = \sup\{d(y, R^m \setminus W_t)\colon y \in A_t\}.$$

Using the fact that A_t is closed in W_t, it is easy to see that for every $t \in \pi(A)$ the set C_t is nonempty and consists precisely of the points from A_t which have maximal distance to $R^m \setminus W_t$ among all points of A_t. Moreover C_t is closed and bounded for every $t \in \pi(A)$. We obtain the definable map $g\colon \pi(A) \to R^m$, $t \mapsto \operatorname{lexmin}(C_t)$ with $(g(t), t) \in A$ for every $t \in \pi(A)$. □

To establish Proposition 5.1 we can essentially follow the proof of [10, Lemma 5.1].

Proof of Proposition 5.1. Clearly (*ii*) implies (*i*). So assume (*i*) holds. By Lemma 5.2 it suffices to show that every D_Σ function $f = (f_1, \dots, f_n)\colon U \to R^m$ is locally affine on a dense subset $V \subseteq U$, as V will automatically be open. In order to show that f is locally affine on a dense subset of U it is enough to construct an open box $B \subseteq U$ restricted to which f is affine. Note that we may assume $n = 1$, as we can first find an open box B_1 such that f_1 is affine on B_1, then find an open box inside B_1, restricted to which f_2 is affine and so on. The open box B_n we end up with, is the desired open box, restricted to which f is affine. We apply induction on m to construct such an open box. $m = 1$ is immediate.

Let $m > 1$. By Theorem 2.15 there is an open box $B = I_1 \times \cdots \times I_m$ in U restricted to which f is continuous and let $B' = I_1 \times \cdots \times I_{m-1}$. For every $x \in B'$ set $f_x\colon I_m \to R$, $x \mapsto f(x,t)$. For $\delta > 0$ let

$$E_\delta = \{(x,t) \in B' \times I_m \colon (t-\delta, t+\delta) \text{ is a subset of } I_m \text{ on which } f_x \text{ is affine } \}.$$

We will now show that $(E_\delta)_{\delta>0}$ is a decreasing definable family of bounded sets each of which is closed in B. Clearly the family is decreasing. Writing $I_m = (a,b)$ we see that for every $\delta > 0$

$$\begin{aligned}\{(x,t) \in B' \times [a+\delta, b-\delta] &\colon \text{for all } y < z \in (-\delta,\delta) \text{ and for all } \epsilon \in (-\delta - y, \delta - z) \\ &\text{it holds } f_x(t+y+\epsilon) - f_x(t+y) = f_x(t+z+\epsilon) - f_x(t+z)\}\end{aligned}$$

equals E_δ, as (x,t) is in the displayed set if and only if f_x is locally affine on $(t-\delta, t+\delta)$, which, by Lemma 5.3, is equivalent to f_x being affine on $(t-\delta, t+\delta)$. This also shows that the continuity of f implies that E_δ is closed.

Let $E = \bigcup_{\delta>0} E_\delta$ and note that $(x,t) \in E$ if and only if f_x is locally affine in t. Since $E = B \cap \bigcup_{\delta>0} \mathrm{cl}(E_\delta)$, it is a D_Σ set such that for every $x \in B'$ the fibre E_x is a dense open subset of I_m. Fact 1.12 implies that E has interior, so after possibly shrinking B, we may assume that for every $x \in B'$, f_x is locally affine on I_m and hence affine on I_m by Lemma 5.3. Therefore there are functions $\alpha, \beta\colon B' \to R$ with $f_x(t) = \alpha(x)t + \beta(x)$ for all $(x,t) \in B$.

We will now show that α is constant. Suppose not. Let $y, y' \in I_m$, with $d = y' - y > 0$. For every $x \in B'$ let $\tilde{\alpha}(x) = f_x(y') - f_x(y) = \alpha(x)d$. The function $\tilde{\alpha}$ is definable and continuous and is constant if and only if α is constant. By Fact 1.2 there is an open interval L contained in the image of $\tilde{\alpha}$. Let $A \subseteq R^m \times R$ be the graph of $\tilde{\alpha}$ restricted to the open definable set $\tilde{\alpha}^{-1}(L)$. Continuity of $\tilde{\alpha}$ ensures that A_r is locally closed for every $r \in L$, so by Lemma 5.4 there is a definable function $g\colon L \to R^m$ with $\tilde{\alpha} \circ g = \mathrm{id}_L$. In particular $f_{g(r)}$ has slope $\frac{r}{d}$ for every $r \in L$. Choose $s \in I_m$ and $s' \in L$. Then there is $\delta > 0$ such that $(s-\delta, s+\delta) \subseteq I_m$ and $(s'-\delta, s'+\delta) \subseteq L$. Define

$$h\colon (s-\delta, s+\delta) \to R,\ t \mapsto f_{g(t+s'-s)}(t) - f_{g(t+s'-s)}(s).$$

For every $t \in (s-\delta, s+\delta)$ we have $h(t) = \frac{1}{d}(t^2 + t(s'-2s) - s(s'-s))$, so h is a continuous definable function which is nowhere locally affine. This contradicts (i), so α has to be constant.

Let $a \in R$ with $\alpha(x) = a$ for all $x \in B'$ and choose $t \in I_m$. Then $\beta(x) = f_x(t) - at$, for every $x \in B'$, so β is definable and continuous. By the inductive assumption there is an open box $B'' \subseteq B'$ restricted to which β is affine. Thus f is affine on $B'' \times I_m$. □

5.2 Defining a field

We will now demonstrate how to define a field from a continuous definable function $I \to R$, that is not generically locally affine. More precisely we will prove:

Theorem 5.5. *Let $\mathcal{R}$ be type A. Exactly one of the following statements holds:*

(a) $\mathcal{R}$ is field-type,

(b) Every definable continuous function $f\colon I \to R$ is generically locally affine on I.

We first establish some results used in the proof of Theorem 5.5.

Lemma 5.6 ([10, Lemma 3.5]). *Let $f\colon [0,b) \to R$ be a definable C^2 function, $X \subseteq R^n$ a definable set and $(g_t\colon I_t \to R)_{t\in X}$ a definable family of C^2 functions such that*

(1) $f'_+(0) = 0$ and $f'(x) > 0$ for all $x \in (0,b)$,

(2) I_t is an open interval containing 0 and $g_t(0) = 0$ for all $t \in X$.

Then the relations $g'_t(0) < g'_s(0)$, $g'_t(0) \leq g'_s(0)$ and $g'_t(0) = g'_s(0)$ are definable on X.

Proof. The proof given in [10] goes through word for word. □

Proposition 5.7. *If $\mathcal{R}$ defines a continuous function $f\colon I \to R$, that is not generically locally affine, then for every $k \geq 3$ there is $b > 0$ and a definable C^k function $g\colon (0,b) \to R$ such that its derivative $g'\colon (0,b) \to (0,1)$ is increasing and bijective.*

Remark. The preceding proposition is more or less straightforward in case R equals $\mathbb{R}$ or more generally in case that R is an archimedean field. Namely by Theorem 4.1 we can assume without loss of generality that f is C^k. Moreover by reparameterizing f we may assume that f' is increasing. Choose $a, b \in I$ with $a < b$ and $f(a)$ rational. Since R is archimedean there is $c \in \mathbb{N}$ with $c(f'(b) - f'(a)) > 1$. By replacing f with the definable cf and decreasing b, we can assume that $f'(b) - f'(a) = 1$. Finally the function $x \mapsto f(a)x$ is definable, as $f(a)$ is rational, so defining g to be $x \mapsto f(x) - f(a)x$ gives a function $(a,b) \to R$ with $g'(a) = 0$ and $g'(b) = 1$.

To establish Proposition 5.7 we need a couple of auxiliary results.

Lemma 5.8. *Let $k \geq 3$. If $\mathcal{R}$ defines a continuous function $f\colon I \to R$ which is not generically locally affine, then $\mathcal{R}$ also defines a C^k function $g\colon (0,b) \to R$ with $b > 0$ such that*

(1) $g'(x) > 0$ for all $x \in (0,b)$ and $\lim_{x\to b} g'(x) = \infty$,

(2) g' is strictly increasing.

Proof. By Theorem 4.1 there is a dense open subset U of I on which f is C^k. As f is not generically locally affine, there has to be $x_0 \in U$ with $f''(x_0) \neq 0$. By possibly replacing f with $-f$ we may assume that $f''(x_0) > 0$. Continuity of f'' gives us an open interval $J = (c,d) \subseteq U$ with $x_0 \in J$ and $f''(x) > 0$ for all $x \in J$. Fact 1.10 implies that f' is strictly increasing on J, so after possibly shrinking J we may assume that f' has either positive only or negative only values on J. In the latter case we replace f by $x \mapsto -f(c+d-x)$. Finally we can shift f to assume that $c < 0 < d$.

Let $h\colon (\frac{c}{2}, \frac{d}{2}) \to R$ be the definable C^k function with $h(x) = \frac{1}{2}f(2x) - f(x)$. Then $h'(0) = 0$ and $h''(0) > 0$. By decreasing d we can assume that $h''(x) > 0$ for all $x \in (0, \frac{d}{2})$, which in turn implies $h'(x) > 0$ for all $x \in (0, \frac{d}{2})$. Let $b' = \frac{d}{2}$ and replace h by definable map $(0,b') \to R$, $x \mapsto h(x) - h(0)$. With $b = h(b')$, h is a definable increasing bijection $(0,b') \to (0,b)$. Let $\tilde{g}\colon (0,b) \to (0,b')$ be the definable C^k inverse of h. The Chain Rule of differentiation implies that

$$\tilde{g}'(x) = \frac{1}{h'(\tilde{g}(x))} > 0,\ \forall x \in (0,b),$$

so $\tilde{g}'$ is strictly decreasing and $\lim_{x\to 0} \tilde{g}'(x) = \infty$. Therefore

$$g\colon (0,b) \to R,\ x \mapsto -\tilde{g}(b-x)$$

is the desired function. □

Lemma 5.9. *If $k \geq 3$ and $\mathcal{R}$ defines a function $f\colon [0,c] \to R$ with $c > 0$ that is C^k on $(0,c)$ such that*

(1) $f'(x) > 0$, for all $x \in (0,c)$ and $\lim_{x\to 0} f'(x) = 0$,

(2) $f''(x) > 0$, for all $x \in (0,c)$,

(3) $f'''(x) \leq 0$, for all $x \in (0,c)$,

then $\mathcal{R}$ also defines a C^k function $g\colon (0,b) \to R$ with $b > 0$ such that its derivative $g'\colon (0,b) \to (0,1)$ is increasing and bijective.

Proof. In a first step we will construct a definable map $h\colon [0,c] \to R$, which is C^k on $(0,c)$ with the properties

(1) $h'(x) > 0$, for all $x \in (0,c)$ and $\lim_{x\to c} h'(x) = \infty$,

(2) $h''(x) > 0$, for all $x \in (0,c)$,

(3) h'' is strictly increasing on $(0, c)$.

The function f is a strictly increasing continuous definable function, so by replacing f with $x \mapsto f(x) - f(0)$, we may assume that $f\colon [0, c] \to [0, d]$ is bijective, where $d = f(c)$. Let $\tilde{h}\colon [0, d] \to [0, c]$ be its compositional inverse, which is C^k on $(0, d)$ by the Inverse Function Theorem. Using the chain rule of differentiation we see that for every $x \in (0, c)$

$$\begin{aligned}
\tilde{h}'(f(x)) &= \frac{1}{f'(x)} > 0, \\
\tilde{h}''(f(x)) &= -\frac{f''(x)}{f'(x)^3} < 0, \\
\tilde{h}'''(f(x)) &= 3\frac{f''(x)^2}{f'(x)^5} - \frac{f'''(x)}{f'(x)^4} > 0.
\end{aligned}$$

Moreover $\lim_{x\to 0} \tilde{h}'(x) = \infty$. Therefore the chain rule of differentiation implies that $h\colon [0, d] \to R$, $x \mapsto -\tilde{h}(d - x)$ has the desired properties. This finishes the first step.

In the second step we will construct a map g with the desired properties using h. Note that h' is definable in the definably complete structure $\mathcal{R}_*$. This implies that there is $\delta \in (0, \frac{d}{2})$ with $h'(\frac{d}{2} + \delta) - h'(\frac{d}{2}) = 1$. Let $b' = \frac{d}{2} - \delta$ and define

$$\tilde{g}\colon [0, b'] \to R,\ x \mapsto h(x + \delta + \tfrac{d}{2}) - h(x + \tfrac{d}{2}) - h(\delta + \tfrac{d}{2}) + h(\tfrac{d}{2})).$$

By construction we have $\lim_{x\to 0} \tilde{g}'(x) = 1$ and $\lim_{x\to b'} \tilde{g}'(x) = \infty$. Moreover for every $x \in (0, b')$ we have $\tilde{g}''(x) > 0$, as h'' is strictly increasing. This means that $\tilde{g}'$ is a strictly increasing bijection $(0, b') \to (1, \infty)$. Let $b = \tilde{g}(b')$. Note that $\tilde{g}$ is an increasing bijection $[0, b'] \to [0, b]$ which is C^k on $(0, b')$. Let $\tilde{g}^{-1}\colon [0, b] \to [0, b']$ be its compositional inverse, which is C^k on $(0, b)$. Finally it is easy to check that

$$g\colon (0, b) \to R,\ x \mapsto -\tilde{g}^{-1}(b - x)$$

has the desired properties. □

Proof of Proposition 5.7. By Lemma 5.8 there is a definable C^k function $h\colon (0, c) \to R$, with h' strictly increasing, $h'(x) > 0$ for all $x \in (0, c)$ and $\lim_{x\to c} h'(x) = \infty$. We will first show that at least one of the following three statements is true.

(a) h'' is strictly increasing.

(b) h'' is strictly decreasing somewhere, i.e. $h''' < 0$ on some subinterval $J \subseteq (0, c)$.

(c) h'' is constant on some subinterval $J \subseteq (0, c)$.

Consider the continuous map $h'''\colon (0, c) \to R$, which is definable in $\mathcal{R}_*$. If $h'''(x) < 0$ for some $x \in (0, c)$, then there is an open interval $J \subseteq (0, c)$ with $h'''(x) < 0$ for all $x \in J$. So in this case (b) holds true. So suppose that $h'''(x) \geq 0$ for all $x \in (0, c)$. If $\{x \in I\colon h'''(x) = 0\}$ has interior, (c) holds. If $\{x \in I\colon h'''(x) = 0\}$ does not have interior, Fact 1.10 implies that h'' is strictly increasing.

First suppose that (a) holds. As h' is strictly increasing we know that $h''(x) \geq 0$ for all $x \in (0, c)$. As h'' is strictly increasing we even have $h''(x) > 0$ for all $x \in (0, c)$. The second step in the proof of Lemma 5.9 provides the desired map g.

Suppose (b) holds. By restricting to a smaller closed subinterval of $(0, c)$ and shifting h we may assume that h is C^k on $[0, d]$ and that $h''(0) > 0$ and $h'''(0) < 0$. Let $f\colon [0, d] \to R,\ x \mapsto \frac{1}{2}h(2x) - h(x)$. Using the chain rule and the fact that h is C^k in 0, we calculate $f'(0) = 0$, $f''(0) > 0$ and $f'''(0) < 0$. Decreasing d if necessary, we may assume that for every $x \in (0, d)$ it holds $f''(x) > 0$ and $f'''(0) < 0$. This in turn implies that $f'(x) > 0$ for all $x \in (0, c)$. Now we can apply Lemma 5.9 to find g.

Finally suppose (c) holds. Let $J = (a, b) \subseteq (0, c)$ be some interval with $h'''(x) = 0$ for all $x \in I$. Applying the Theorem on Constants three times shows that h is a polynomial of degree at most 2 on J. As h' is strictly increasing on $(0, c)$, it has to be a polynomial of degree exactly 2. This also implies that $h''(x) > 0$ for all $x \in J$. It is easy to check that

$$f\colon (0, \tfrac{b-a}{2}) \to R,\ x \mapsto \tfrac{1}{2}h(2x + a) - h(x + a)$$

satisfies $f'(x) > 0$, $f''(x) > 0$ and $f''' = 0$ for all $x \in (0, \frac{b-a}{2}$ as well as $\lim_{x \to 0} f'(x) = 0$. Again Lemma 5.9 provides the desired function g. □

We can now prove Theorem 5.5 by the ideas presented in [10].

Proof of Theorem 5.5. We begin by proving that (a) and (b) are exclusive. So suppose that $\mathcal{R}$ is field-type, that is there is an open bounded interval I and two definable functions $\oplus, \otimes\colon I^2 \to I$ such that $(I, <, \oplus, \otimes)$ is an ordered field isomorphic to $(R, <, +, *)$. It is easy to see that $\oplus$ and $\otimes$ are continuous maps. Therefore

$$f\colon I^3 \to I,\ (x, y, z) \mapsto (x \otimes z) \oplus y$$

is a definable continuous map. Once we show that f is not generically locally affine, Theorem 5.1 shows that there is a continuous definable map $J \to R$ which is not generically locally affine. Towards a contradiction assume that f is generically locally affine. By restricting to an open definable subset U of I^3 we may assume that f is affine. So there are linear maps $h_x, h_y, h_z\colon R \to R$ and $b \in R$ such that for all $(x, y, z) \in U$ we have

$$f(x, y, z) = h_x(x) + h_y(y) + h_z(z) + b.$$

As f locally varies in x, y and z, h_x, h_y and h_z are all non constant. This implies that we find $(x, y), (x', y') \in R^2$ with $x \neq x'$ and $h_x(x) + h_y(y) = h_x(x') + h_y(y')$ such that $U_{(x,y)}$ and $U_{(x',y')}$ are non empty respectively. So for every $z \in U_{(x,y)} \cap U_{(x',y')}$ we have $(x \otimes y) \oplus z = (x' \otimes y') \oplus z$. Using that $(I, \oplus, \otimes)$ is a field, we see that there can at most be one $z \in I$ with this property, contradicting the fact that $U_{(x,y)} \cap U_{(x',y')}$ is open and non empty. This shows that f is not locally affine, so (a) and (b) are indeed exclusive.

We finish the proof by demonstrating that whenever $\mathcal{R}$ defines a continuous function $I \to R$ which is not locally affine, it has to be field-type. Let $k \geq 3$. By Proposition 5.9 there is $b > 0$ and a continuous C^k function $g\colon (0,b) \to R$ with $g'\colon (0,b) \to (0,1)$ strictly increasing and bijective. The Mean Value Theorem implies that g is bounded and by the Intermediate Value Theorem the image of g is an interval. Moreover the interval is necessarily open. Set $f\colon [0,\frac{b}{2}) \to R$, $x \mapsto \frac{1}{2}g(\frac{b}{2}+2x) - f(\frac{b}{2}+x)$ and let $F = (-2b, 2b)$. One easily checks that f is C^1 as well as

(1) $f'_+(0) = 0$,

(2) $f'(x) > 0$ for all $x \in (0,\frac{b}{2})$.

We will now construct a definable family of functions indexed by F. For $t \in (0,b)$ let $I_t = (-t, b-t) \subseteq R$ and $g_t\colon I_t \to R$, $x \mapsto g(t+x) - g(t)$. Note that g_t is strictly increasing and C^k and that its image is a bounded open interval, say J_t. If $t \in (b, 2b)$ we let $g_t\colon J_{2b-t} \to R$ be the compositional inverse of g_{2b-t}. If $t \in (-b, 0)$, let $g_t\colon I_{-t} \to R$, $x \mapsto -g_{-t}(x)$, with image $-J_{-t}$. If $t \in (-2b,-b)$, let $g_t\colon -J_{2b+t} \to R$ be the compositional inverse of $g_{-(2b+t)}$. Finally let $g_0\colon R \to R$, $x \mapsto 0$, $g_b\colon R \to R$, $x \mapsto x$ and $g_{-b}\colon R \to R$, $x \mapsto -x$. We have constructed a definable family of C^k functions $(g_t\colon I_t \to R)_{t\in F}$ with the following properties:

(1) For every $x \in R$ there is a unique $t \in F$ with $g'_t(0) = x$,

(2) if $s, t \in F$, then $s < t$ if and only if $g'_s(0) < f'_t(0)$,

(3) $g_t(0) = 0$ for all $t \in F$,

(4) I_t is an open neighbourhood of 0 for every $t \in F$.

Now we define $\oplus$ and $\otimes$ on F. For $s, t \in F$ we set $s \oplus t$ to be the unique element of F with

$$g'_{s\oplus t}(0) = (g_s + g_t)'(0)$$

and we set $s \otimes t$ to be the unique element of F with

$$g'_{s\otimes t}(0) = (g_s \circ g_t)'(0).$$

(Note that $g_s \circ g_t$ is defined on some open interval around 0.) Using Lemma 5.6 together with f it follows, that $\oplus$ and $\otimes$ are definable functions $F^2 \to F$. The construction guarantees that for all $s, t \in F$

$$g'_{s\oplus t}(0) = g'_s(0) + g'_t(0) \text{ and } g'_{s\otimes t} = g'_s(0)g'_t(0).$$

So the mapping $t \mapsto g'_t(0)$ is an isomorphism $\tau\colon (F, <, \oplus, \otimes) \to (R, <, +, *)$. Since $g'\colon (0,b) \to (0,1)$ is C^{k-1} and bijective, the Inverse Function Theorem implies that it is a C^{k-1} diffeomorphism, so τ viewed as a map $F \setminus \{-b, 0, b\} \to R \setminus \{-1, 0, 1\}$ is a C^k diffeomorphism as well. $\square$

Chapter 6

Generic differentiability in several variables

In this last chapter we want to use Theorem B to prove, that every D_Σ function $U \to R^n$, with $U \subseteq R^m$ open is generically C^k, in doing so obtaining a strengthened multivariable version of Theorem 4.1.

Theorem C. *Let $\mathcal{R}$ be type A. Every D_Σ function $f\colon U \to R^n$, where $U \subseteq R^m$ is open, is generically C^k.*

To establish Theorem C we will essentially follow the proof of [10, Theorem B], which roughly goes as follows. By Theorem B of this thesis, we only need to consider type A structures, which are field-type, as generically locally affine functions are generically locally C^∞. Lemma 6.1 allows us to reduce the problem to type A expansions of $(R, <, +, *)$. But as we will see in Lemma 6.2, those expansions have to be restrained, so the desired result follows from Fact 4.2.

Lemma 6.1 ([10, Lemma 6.2])**.** *Let $k \geq 2$ and suppose that $\mathcal{R}$ is field-type. Then there is an open interval I, definable functions $\oplus, \otimes\colon I^2 \to I$ an isomorphism $\tau\colon (I, <, \oplus, \otimes) \to (R, <, +, *)$ and an open interval $J \subseteq I$ such that the restriction of τ to J is a C^k-diffeomorphism $J \to \tau(J)$, with $\tau(J) \subseteq R$ open.*

Proof. If $\mathcal{R}$ is field-type, for a given $k \geq 2$ the proof of Theorem 5.5 constructs $I = (-2b, 2b)$ for some $b > 0$ and definable $\oplus, \otimes\colon I^2 \to I$ such that the isomorphism $\tau\colon (I, <, \oplus, \otimes) \to (R, <, +, *)$ restricts to a C^{k-1}-diffeomorphism $(0, b) \to (0, 1)$. $\square$

Lemma 6.2 ([10, Lemma 6.3])**.** *If $\mathcal{R} = (R, <, +, *, \dots)$ is type A, every D_Σ function $U \to R^m$, with $U \subseteq R^n$ open, is generically C^k.*

Proof. Note that $\mathcal{R}$ is restrained. Namely suppose $\mathcal{R}$ were unrestrained. Then there is a definable surjection $g\colon N \mapsto Q$ by Fact 2.4. For $t > 0$ define

$$X_t = \{g(n) \colon n \in N_{\leq t}\}.$$

Definable induction shows that for every $n \in N$, X_n is nowhere dense, in particular $(X_t)_{t>0}$ is an increasing family of nowhere dense sets, whose union is Q. By Lemma

2.9, Q is dense in R, so this contradicts the SBCT, which holds in all type A expansions. Thus $\mathcal{R}$ cannot be unrestrained.

Write $f = (f_1, \ldots, f_n)$, with $f_i \colon U \to R$ for $i = 1, \ldots, n$. Each f_i is a D_Σ function, so generically continuous. Fact 4.2 implies that for every k, each f_i is generically C^k. This implies that for every k, f is generically C^k. $\square$

Proof of Theorem C. Recall that we only need to consider type A structures, which are field-type.

Let $k \geq 2$ and $I, J, \oplus, \otimes$ and $\tau \colon I \to R$ as described in Lemma 6.1. Let $U \subseteq R^m$ be open and $f \colon U \to R^n$ be a D_Σ function. We will show, that f is generically C^k. By Theorem 2.15, we may assume that f is continuous. Let $\mathcal{I}$ be the expansion of $(I, <, \oplus, \otimes)$ by all $\mathcal{R}$-definable sets $A \subseteq I^i$, for $i \in \mathbb{N}$. Note that $\mathcal{I}$ is a definably complete expansion of an ordered field. Moreover $\mathcal{I}$ is type A. Fix $x_0 \in J^m$. For $x \in U$ let

$$f_x \colon (x_0 - x + U) \cap J^m \to R^n,\ y \mapsto f(y + x - x_0).$$

For every $x \in U$, $V_x = f_x^{-1}(J^n) \subseteq J^m$ is an open neighbourhood of x_0 and $f_x(V_x) \subseteq J^n$. Thus $f_x \colon V_x \to J^n$ is $\mathcal{I}$-definable for every $x \in U$, so Lemma 6.2 implies, that f_x is $\mathcal{I}$-generically C^k with respect to the field structure of $\mathcal{I}$. As $\tau \colon J \to \tau(J)$ is a C^k-diffeomorphism, f_x is C^k in $y \in V_x$ with respect to the field structure of $\mathcal{I}$ if and only if f_x is C^k in y with respect to the field structure of $\mathcal{R}_*$. Namely define

$$\tau_i \colon J^i \to \tau(J)^i,\ (x_1, \ldots x_i) \mapsto (\tau(x_1), \ldots, \tau(x_i))$$

for $i \in \mathbb{N}$. f_x is C^k with respect to the field structure of $\mathcal{I}$ if and only if $\tau_n \circ f_x \circ \tau_m^{-1}$ is C^k with respect to the field structure of $\mathcal{R}_*$. All $\tau_i \colon J^i \to \tau(J)^i$ are C^k-diffeomorphisms respectively, so this is equivalent to f_x being C^k with respect to the field structure of $\mathcal{R}_*$.

Since $\oplus$ and $\otimes$ are $\mathcal{R}$-definable, f_x is $\mathcal{R}$-generically C^k with respect to the field structure of $\mathcal{R}_*$. Moreover for every $y \in V_x$, f_x is C^k in y if and only if f is C^k in $y - x_0 + x$. This demonstrates that f is generically C^k on U. $\square$

Bibliography

[1] R. P. Boas Jr. and D. V. Widder. "Functions with positive differences". In: *Duke Mathematical Journal* 7.1 (1940), pp. 496–503.

[2] A. Dolich, C. Miller, and C. Steinhorn. "Structures having o-minimal open core". In: *Transactions of the American Mathematical Society* 362.3 (2010), pp. 1371–1411.

[3] L. v. d. Dries. *Tame Topology and o-minimal structures*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998.

[4] A. Fornasiero. "A Note on Hieronymi's Theorem: Every definably complete structure is definably baire". In: *Groups, Modules, and Model Theory - Surveys and Recent Developments : In Memory of Rüdiger Göbel*. Ed. by Manfred Droste et al. Springer International Publishing, 2017, pp. 301–315.

[5] A. Fornasiero. "Locally o-minimal structures and structures with locally o-minimal open core". In: *Annals of Pure and Applied Logic* 164.3 (2013), pp. 211–229. ISSN: 0168-0072.

[6] A. Fornasiero. "Tame structures and open cores". In: (2010). arXiv: `1003.3557`.

[7] A. Fornasiero and P. Hieronymi. "A fundamental dichotomy for definably complete expansions of ordered fields". In: *The Journal of Symbolic Logic* 80.4 (2015), pp. 1091–1115.

[8] A. Fornasiero, P. Hieronymi, and E. Walsberg. "How to avoid a compact set". In: *Advances in Mathematics* 317 (2017), pp. 758–785.

[9] P. Hieronymi. "An analogue of the Baire Category Theorem". In: *The Journal of Symbolic Logic* 78.1 (2013), pp. 207–213. (Visited on 08/19/2023).

[10] P. Hieronymi and E. Walsberg. "A tetrachotomy for expansions of the real ordered additive group". In: *Selecta Mathematica, New Series* 27 (2021).

[11] P. Hieronymi and E. Walsberg. "Interpreting the monadic second order theory of one successor in expansions of the real line". In: *Israel Journal of Mathematics* 224 (2018), pp. 39–55.

[12] E. Hrushovski. "A new strongly minimal set". In: *Annals of Pure and Applied Logic* 62.2 (1993), pp. 147–166.

[13] J. L. Kelley. *General Topology*. Graduate Texts in Mathematics. Springer New York, 1975.

[14] K. Königsberger. *Analysis 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2003.

[15] C. Miller. "Expansions of dense linear orders with the Intermediate Value Property". In: *The Journal of Symbolic Logic* 66.4 (2001), pp. 1783–1790.

[16] B. I. Zil'ber. "Strongly minimal countably categorical theories. II". In: *Siberian Mathematical Journal* 25.3 (1984), pp. 396–412.