Homework problems (due May 29)

Problem 1 (Plane quadrics)

(a) Let $\mathcal{L}$ be a line bundle on the curve C/k. Assume that $\deg(\mathcal{L}) = 1$, that $\mathcal{L}$ is globally generated, and that $H^0(C, \mathcal{L})$ is 2-dimensional as k-vector space with basis x, y. Show that x, y define an isomorphism

$$[x : y] : C \xrightarrow{\sim} \mathbb{P}^1_k.$$

(b) Let C/k be a curve of genus 0. Show that the following three statements are equivalent:

(1) The set of rational points $C(k)$ is non-empty.
(2) There exists a line bundle of degree 1 on C.
(3) There exists an isomorphism $C \xrightarrow{\sim} \mathbb{P}^1_k$.

(c) Prove that $V_+(X^2 + Y^2 + Z^2) \subseteq \mathbb{P}^2_{\mathbb{R}}$ is a curve of genus 0 that is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$.

Problem 2 (Ample line bundles)

(a) Show that a line bundle $\mathcal{L}$ on a curve C is ample if and only if $\deg(\mathcal{L}) > 0$.

Hint: Recall from Algebraic Geometry 1, §25 and §26, that a line bundle $\mathcal{M}$ on a finite type scheme over a field is ample if and only if some power of it defines an immersion into projective space.

(b) Let $x \in C$ be a closed point. Prove that the complement $C \setminus \{x\}$ is affine.