Exercise Sheet 4

Discussed on 05.05.2021

Problem 1. Let k be an algebraically closed field with $\operatorname{char} k \neq 2$ and let $\lambda \in k$ be any element with $\lambda \neq 0, 1$. Let E be the smooth projective curve

$$E = V(y^2 z - x(x-z)(x-z\lambda)) \subset \mathbb{P}^2_k.$$

(a) Show that the projection

$$E^0 := V(y^2 - x(x-1)(x-\lambda)) \subset \mathbb{A}^2_k \to \mathbb{A}^1_k$$

to the y-axis extends uniquely to a map $\mathrm{pr}_y \colon E \to \mathbb{P}^1_k$. Compute $\deg \mathrm{pr}_y$.

(b) For every closed point $w \in E$, compute the ramification e_w of pr_y at w. Can you do this explicitly using the definition of e_w?

(c) Verify the Riemann-Hurwitz formula for pr_y.

Problem 2. Let k be a field of characteristic 0 and let X and Y be smooth proper connected curves over k.

(a) If a non-constant k-morphism $f \colon X \to \mathbb{P}^1_k$ is not an isomorphism then it ramifies at some point of X. If k is algebraically closed, then there are at least two points in X at which f ramifies.

(b) If X and Y are elliptic curves, show that every non-constant k-morphism $X \to Y$ is unramified.

Problem 3. *The following problem has already been on the last sheet, but has not been discussed yet.*

(a) Let E be an elliptic curve over $\mathbb{C}$. Show that for every $N > 0$, $E[N] := \ker([N] \colon E \to E)$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$.

(b) A *level N-structure* on E is an isomorphism $\alpha \colon (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$. A morphism $(E, \alpha) \to (E', \alpha')$ of elliptic curves with level N-structures is a morphism $f \colon E \to E'$ of elliptic curves such that $\alpha' = f \circ \alpha$.

Let $\Gamma(N) \subset \mathrm{GL}_2(\mathbb{Z})$ be the kernel of the projection $\mathrm{GL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Show that there is a canonical bijection

$$\Gamma(N) \setminus \mathcal{H}^{\pm} \overset{\sim}{\longleftrightarrow} \{\text{elliptic curves}/\mathbb{C} \text{ with level } N\text{-structure}\}/\cong$$

(c) Show that for $N \geq 4$, the action of $\Gamma(N)$ on $\mathcal{H}^{\pm}$ is free, i.e. all stabilizers are trivial.