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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 8

Exercise 38. (Going-up property, 3 points)
One says that a ring homomorphism $f\colon A \to B$ satisfies the *going-up property* if for a prime ideals $\mathfrak{q} \subset B$ and $\mathfrak{p} := \mathfrak{q}^c \subset \mathfrak{p}' \subset A$, there exists a prime ideal $\mathfrak{q} \subset \mathfrak{q}' \subset B$ with $\mathfrak{q}'^c = \mathfrak{p}'$.

Assume B is Noetherian. Show that f has the going-up property if and only if the induced map $\varphi\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is closed (i.e. the image of a closed set is again closed).
Hint: One might need (to prove) the following property: Let $\mathfrak{p} \subset A$ be a prime ideal then the closure $\overline{\{\mathfrak{p}\}}$ is $V(\mathfrak{p})$.

Exercise 39. (Cusp, 4 points)
Discuss the ring $A = k[x, y]/(y^2 - x^3)$ similarly to the discussion of $k[x, y]/(y^2 - x^2(x + 1))$ in Section 10.4. Do we have, in this case, $A \cong k[t]$?

Exercise 40. (Ring of invariants, 3 points)
Let A be a ring and G a finite group of ring automorphisms $g\colon A \to A$. Define the *ring of G-invariants* $A^G := \{a \in A \mid \forall g \in G : g(a) = a\}$ and $\varphi : \operatorname{Spec}(A) \to \operatorname{Spec}(A^G)$ the continuous map induced by the natural inclusion $i : A^G \hookrightarrow A$.

1. Show that A is integral over A^G.
2. Let $\mathfrak{p} \subset A^G$ be a prime ideal. Prove that G acts transitively on $\varphi^{-1}(\mathfrak{p})$ (as a consequence $\varphi^{-1}(\mathfrak{p})$ is finite).
   Hint: One might need (to prove) the following result: let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ be prime ideals and $\mathfrak{a} \subset A$ an ideal satisfying $\mathfrak{a} \not\subset \mathfrak{p}_i$ for any i. Then $\mathfrak{a} \not\subset \cup_{i=1}^n \mathfrak{p}_i$.

Exercise 41. (Circle as a spectrum, 4 points)
Consider the ring $A := k[x, y]/(x^2 + y^2 - 1)$. Show that it is factorial for $k = \mathbb{C}$ and not factorial for $k = \mathbb{R}$. (Observe that in the two cases A is isomorphic to the ring of functions $\mathbb{C}[e^{it}, e^{-it}]$ resp. $\mathbb{R}[\sin(t), \cos(t)]$.)

Exercise 42. (Extending ring homomorphisms into fields, 3 points)
Let A be a subring of B such that B is integral over A. Show that every ring homomorphism $f\colon A \to K$ with K an algebraically closed field can be extended to a ring homomorphism $\tilde{f}\colon B \to K$.
Hint: Reduce to the case A integral and get some inspiration from a similar fact for field extensions.

Solutions to be handed in before Tuesday June 2, 4pm.