Prof. Dr. Daniel Huybrechts
Dr. René Mboro

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Exercises, Algebra I (Commutative Algebra) – Week 6

Exercise 27. (Basic open sets, 3 pts)
Let $a \in A \setminus \mathfrak{N}$. Show that the map $\varphi\colon \mathrm{Spec}(A_a) \to \mathrm{Spec}(A)$ induced by the natural ring homomorphism $f\colon A \to A_a$ describes a homeomorphism $\psi\colon \mathrm{Spec}(A_a) \to D(a)$ (i.e. ψ and ψ^{-1} are both bijective and continuous).

Exercise 28. (Consecutive localization, 2 points)
Let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset A$ be two prime ideals. Show that the localization of $A_{\mathfrak{p}_2}$ in the prime ideal corresponding to $\mathfrak{p}_1$ is isomorphic to $A_{\mathfrak{p}_1}$.

Exercise 29. (Comparing basic open sets, 3 points)
Let A be a ring and $a, b \in A \setminus \mathfrak{N}$. Show that $D(a) \subset D(b)$ if and only if $\frac{b}{1} \in A_a$ is a unit. Furthermore, show that in this case the natural ring homomorphism $A \to A_a$ factorizes via a ring homomorphism $A_b \to A_a$. Conclude from this that $D(a) = D(b)$ if and only if $A_a \cong A_b$.

Exercise 30. (Disconnected Spec(A) and idempotents, 4 points)
A topological space X is called *disconnected* if X is the disjoint union of two non-empty open subsets (or, equivalently, of two non-empty closed subsets).

Show that Spec(A) with the Zariski topology is disconnected if and only if there exists an element $0, 1 \neq e \in A$ with $e^2 = e$. (Such an element is called idempotent.) *Hint:* For the if' consider $e' := 1 - e$ and observe $e \cdot e' = 0$. For the 'only if' use the standard properties of $V(\mathfrak{a})$ and the description of the nilradical to construct idempotents in this way.

Exercise 31. (Irreducible Spec(A), 2 points)
A topological space X is called *irreducible* if X is non-empty and the intersection $U \cap V$ of any two non-empty open subsets $U, V \subset X$ is again non-empty. Equivalently, X is not the union of two proper closed subsets.

Show that Spec(A) with the Zariski topology is irreducible if and only if the nilradical $\mathfrak{N} \subset A$ is a prime ideal.

Exercise 32. (Idempotent ideals, 5 points)
Show that for an ideal $\mathfrak{a} \subset A$ the following conditions are equivalent:

(i) $A/\mathfrak{a}$ is a projective A-module.

(ii) $A/\mathfrak{a}$ is a flat A-module and $\mathfrak{a}$ is finitely generated.

(iii) $\mathfrak{a}$ is finite and idempotent (i.e. $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$).

(iv) $\mathfrak{a} = (e)$ for some idempotent e.

(v) $\mathfrak{a}$ is a direct summand of A.

Please turn over.

Solutions to be handed in before Monday May 18, 4pm.

For your convenience we collect a few standard facts concerning tensor products. You may want to revise the arguments how to prove those.

1. Let A be a ring and M, N, and P be A-modules. Then there exist natural isomorphisms

$$M \otimes_A N \cong N \otimes_A M \text{ and } (M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P).$$

2. Let $f\colon A \to B$ be a ring homomorphism and M an A-module and N a B-module. Then there is natural isomorphism

$$M \otimes_A N \cong M \otimes_A B \otimes_B N.$$

3. Let $f\colon A \to B$ be a ring homomorphism and M and N be A-modules. Then there are natural isomorphisms

$$(M \otimes_A N) \otimes_A B \cong (B \otimes_A M) \otimes_A N \cong M \otimes_A (N \otimes_A B) \cong (M \otimes_A B) \otimes_B (N \otimes_A B).$$

4. For an A-module M and an ideal $\mathfrak{a} \subset A$, there is a natural isomorphism

$$M \otimes_A A/\mathfrak{a} \cong M/\mathfrak{a}M,$$

where $\mathfrak{a}M \subset M$ is the submodule generated by elements of the form am with $a \in \mathfrak{a}$ and $m \in M$.

5. Let $\mathfrak{a}, \mathfrak{b} \subset A$ be two ideals of A. Show that there is an isomorphism of rings

$$A/\mathfrak{a} \otimes_A A/\mathfrak{b} \cong A/(\mathfrak{a} + \mathfrak{b}).$$

6. For an A-module M and a multiplicative set $S \subset A$ there exists a natural isomorphism

$$M \otimes_A S^{-1}A \cong S^{-1}M.$$