By hypothesis, $a_p \leqslant b_p$. That is, the first multiple of p in the sequence $a, a-1, \dots, 1$ will occur not later than the first multiple of p in the sequence $b, b-1, \dots, b-a+1$. Thus $\alpha(p) \geqslant \beta(p)$. But if $p > a$, then $\alpha(p) = 0$. So, $\beta(p) = 0$ also, and neither A nor B is divisible by p.

We have

$$\binom{b}{a} = \frac{B}{A} = \prod_{p \leqslant a} p^{\sum_{k=1}^{\infty} \beta(p^k) - \sum_{k=1}^{\infty} \alpha(p^k)}.$$

Denoting by $\kappa(p)$ the exponent of the highest power of p for which $\beta(p^k) > 0$ we get

$$\sum_{k=1}^{\infty} (\beta(p^k) - \alpha(p^k)) = \beta(p) - \alpha(p) + \sum_{k=2}^{\kappa(p)} (\beta(p^k) - \alpha(p^k))$$

$$- \sum_{k=\kappa(p)+1}^{\infty} \alpha(p^k) \leqslant \sum_{k=2}^{\kappa(p)} 1 = \kappa(p) - 1.$$

Therefore

$$\frac{B}{A} \Big| \prod_{p \leqslant a} p^{\kappa(p)-1},$$

or put in another way

$$\frac{(b-a+1)\cdots(b-1)b}{\prod_{p \leqslant a} p^{\kappa(p)}} \Bigg| \frac{1 \cdot 2 \cdots (a-1)a}{\prod_{p \leqslant a} p}.$$

Here, after factoring, there remain in the right-hand side exactly $a - \pi(a)$ factors each at most a, and in the left-hand side at least $a - \pi(a)$ such factors which are $\geqslant b - a + 1$. Since $b \geqslant 2a$, $b - a + 1 \geqslant a + 1 > a$, so we have a contradiction.

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THE TEACHING OF MATHEMATICS

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The Classification of 1-Manifolds: A Take-Home Exam

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1. Introduction. An effective and much used method for introducing students to a new mathematical topic (e.g., modern algebra) is to pick some important subtopic (say, groups) and then present a discussion of the simplest or most familiar special case (for example, the integers). Applying this doctrine to topology, it is clear that the study of manifolds is a very central subtopic and the simplest special case is surely that of 1-dimensional manifolds. The "classification theorem" of our title says in effect that 1-manifolds are not only simple but they are also familiar, being in fact nothing more than circles or intervals. The theorem itself is probably no great surprise. It is however important and useful in at least one approach to topology in obtaining some of the deeper results connected with fixed point theory. Further, the proof of the theorem is both instructive and nontrivial. It seemed worthwhile, therefore, for pedagogical reasons to present the detailed treatment which follows.

This exposition was motivated by recent experience in trying to teach courses in algebraic and differential topology at the advanced undergraduate level. These subjects, it seems to me, present some special difficulties not present in other courses. Consider, for example, courses in modern algebra or integration theory or point-set topology. In these areas the subject matter has become quite standardized, there are numerous texts that treat the material, and it is possible to get down to business quite rapidly and start presenting some of the important results.

In the topology courses, on the other hand, it seems that no matter what approach one takes it is necessary to do a fair amount of hard work before one can get to the meat of the subject, and I found many of my students were not prepared for this from their experiences in other courses. The problem becomes particularly acute if one demands the same degree of rigor and precision as in, say, a point-set topology course. On the other hand, if in order to "get somewhere" one takes a more relaxed, informal approach, many students become unsure as to when a careful argument is needed and when a wave of the hands is enough.

I certainly have no ready solution for this dilemma which may well be inherent in the nature of the subject itself, but I do want to propose a way of at least getting off to a good start. My thesis is that the problem of classifying all 1-dimensional manifolds provides an excellent bridge between the pure point-set ideas which the students are presumably already familiar with and the new combinatorial material to which they are being introduced. Further, the result can be derived completely rigorously without taking an inordinate amount of time.

Here are some other reasons for working through the 1-manifold theorem:

1. A number of undergraduate texts present the classifications of 2-manifolds at an early stage. It seems rather natural to do the easier 1-manifold theorem first as a sort of warm-up.

2. The classification theorem represents a typical example of a theorem which adduces a global conclusion from local hypotheses, i.e., knowing what a space looks like in the neighborhood of each of its points enables one to conclude exactly what it is "in the large." Such theorems, of course, are central in many branches of geometry and analysis.

3. The 1-manifold theorem is perhaps not so important in the development of algebraic topology, but it plays an absolutely pivotal role in differential topology as

presented in the famous exposition of Milnor [3] based on the work of Hirsch [2] on the Brouwer fixed-1 point theorem.

4. One might say that 1-manifolds themselves are not very exciting. There are only four connected ones (manifolds in this paper will always include manifolds with boundary) and they are the obvious ones. I claim, however, that the proof of this fact is interesting because it requires the use of the Hausdorff separation axiom and it is precisely at the point where this axiom is used that the combinatorial aspect of the problem becomes apparent. I am referring to Proposition 1 of the presentation to follow, and its corollary.

I should remark that proofs of the 1-manifold theorem in the smooth case are given both by Milnor [3] and by Guillemin and Pollack [1] but both of these proofs make use of differentiability. The proof for the topological case which is presented here is not to my knowledge presented in any text. In fact, I have not been able to find anyone who was able to tell when the theorem was first proved or by whom, and I would be most interested in any information on this matter. In any case I don't imagine any proof of the result can be very different from the one presented here.

I have organized the material in the form of a take-home exam because the topic seems ideally suited for this. In fact, I have tried to arrange things so that a person teaching an undergraduate topology course could use this exposition directly as it stands. I even think the material would be suitable for use following the well-known R. L. Moore method in which all proofs are presented in class by the students. The instructor may wish to conduct a somewhat shorter exercise by considering only the case of manifolds without boundary. I have organized the proofs into a series of lemmas and propositions and have provided hints with the purpose of bringing the work to what I consider the appropriate level for upper division undergraduate math majors. I have not included proofs but would be glad, upon request, to send my own set of answers to anyone interested.

2. The Theorem. It will be assumed in what follows that the reader is familiar with standard point set topology and the elementary topological properties of the real numbers, specifically, that connected subsets are intervals, that homeomorphisms between intervals are monotonic, and that open subsets of the reals are unions of disjoint open intervals.

DEFINITION. A 1-manifold is a second countable Hausdorff topological space X such that

(M) X can be covered by open sets each of which is homeomorphic either to the open interval $\langle 0,1\rangle$ or the half-open interval $[0,1\rangle$. Sets of the first type will be called *O-sets*, of the second type *H-sets*, of either type *I-sets*, and the corresponding homeomorphisms to these intervals will be called *O-charts*, *H-charts*, and *I-charts*, respectively. If X can be covered by O-sets it is a ***manifold without boundary***, otherwise it is a ***manifold with boundary***.

CLASSIFICATION THEOREM. *There are exactly four connected* 1-*manifolds* (*up to homeomorphism*) *and they are given by the following table*:

	Without boundary	With boundary
Compact	a circle	a closed interval
Non-compact	an open interval	a half-open interval

PROPOSITION 0. *Each of the four spaces of the table above is a* 1-*manifold.*

We next want to show the necessity of the Hausdorff Axiom.

EXAMPLE 1. Let $X = \langle 0,1\rangle \cup \{p\}$ where p is a singleton. A basis for the open sets of X are all open sets of $\langle 0,1\rangle$ plus all sets of the form $(U - \{1/2\}) \cup \{p\}$ where U is open in $\langle 0,1\rangle$ and $1/2 \in U$. Prove that X is a T_1-space which satisfies condition (M) but is not Hausdorff.

From here on U and V will stand for I-sets in a 1-manifold and ϕ and ψ will be associated I-charts.

LEMMA. *Suppose $U \cap V$ and $U - V$ are nonempty and let (x_n) be a sequence in $U \cap V$ converging to x in $U - V$. Then the sequence $\psi(x_n)$ has no limit point in $\psi(V)$.*

Hint: Use the Hausdorff property.

We say that U and V ***overlap*** if $U \cap V$, $U - V$ and $V - U$ are nonempty.

DEFINITION. An open subinterval of $\langle 0,1\rangle$ is ***lower*** if it is of the form $\langle 0, b\rangle$ and ***upper*** if it is of the form $\langle a,1\rangle$. A subinterval which is either upper or lower is called ***outer***. It is easy to see that an open interval in $\langle 0,1\rangle$ is outer if and only if it contains a sequence with no limit point in $\langle 0,1\rangle$. Similarly, in $[0,1\rangle$, a subinterval is called ***upper*** and ***outer*** if it is of the form $\langle a,1\rangle$. (There are, by definition, no lower open subintervals of $[0,1\rangle$.) An open subinterval of $[0,1\rangle$ is outer if and only if it contains a sequence with no limit point in $[0,1\rangle$.

PROPOSITION 1. *If U and V overlap and W is a component of $U \cap V$, then $\phi(W)$ and $\psi(W)$ are outer intervals.*

Hint: Note that $\phi(W)$ is a proper subinterval of $\phi(U)$. Using the lemma show that $\phi(W)$ is an open interval. Then construct an appropriate sequence in $\phi(W)$ and use the lemma again.

COROLLARY. *If U and V are I-sets, then $U \cap V$ has at most two components. If either U or V is an H-set, then $U \cap V$ is connected.*

PROPOSITION 2. *If X is connected and $U \cap V$ has two components, then X is a circle.*

Hints: (a) Let the components be Z and W and choose O-mappings ϕ and ψ so that $\phi(W)$ and $\psi(W)$ are lower and $\phi(Z)$ and $\psi(Z)$ are upper.

(b) Let $a = \sup \phi(W)$, $a' = \inf \phi(Z)$, and $b = \sup \psi(W)$, $b' = \inf \psi(Z)$. Let f map $[0,1]$ to the unit square by a piecewise linear mapping with

$$f(0) = (0,0), \quad f(a) = (1,0), \quad f(a') = (1,1), \quad f(1) = (0,1).$$

Let g map $[b, b']$ linearly with $g(b) = (0,0)$, $g(b') = (0,1)$. Define η on $U \cup V$ by $\eta(x) = f \circ \phi(x)$ for $x \in U$ and $\eta(x) = g \circ \psi(x)$ for $x \in V - U$.

(c) Prove that η is a homeomorphism of X onto the unit square using compactness of $U \cup V$ and connectedness of X.

PROPOSITION 3. Hypotheses: *U and V overlap and $U \cap V$ is connected.* Conclusion: (i) *If U and V are O-sets, so is $U \cup V$.*

(ii) *If U is an H-set and V an O-set, then $U \cup V$ is an H-set.*

(iii) *If U and V are H-sets, then $U \cup V = X$ and X is a closed interval.*

Hint: Letting $W = U \cap V$ choose ϕ and ψ so that $\phi(W)$ and $\psi(W)$ are upper and let $b = \inf \psi(W)$; define η on $U \cup V$ by $\eta(x) = \phi(x)$ for $x \in U$ and $\eta(x) = 1 + b - \psi(x)$ for $x \in V - U$.

All right; now give the proof of the Classification Theorem for the compact case. (*Hint*: Use induction on the number of sets in a finite open covering.)

For the noncompact case we must use separability of the space C, for there exist nonseparable Hausdorff 1-manifolds, the so-called Long Lines, which will be discussed in the Appendix.

Assuming second countability, prove the Classification Theorem for the noncompact case. *Hint*: Consider first the case without boundary. Let (U_i), $i = 1, 2, \ldots$ be a countable covering of X by O-sets and define a nested sequence (V_i) of O-sets inductively as follows. $V_1 = U_1$ and $V_{n+1} = V_n \cup U_k$ where k is the smallest subscript such that U_k meets V_n. Prove that $V = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} U_n = X$ (this is the crucial step). Finally, define mappings ψ_n from V_n to R inductively as follows: ψ_1 is an O-mapping of V_1. Now suppose $\psi_n(V_n) = \langle a, b \rangle$. Let $\tilde{\psi}_n$ be an O-mapping on V_{n+1} such that $\tilde{\psi}_n \circ \psi_n^{-1} \langle a, b \rangle = \langle \alpha, \beta \rangle$ and use this to define an extension of ψ_n to a mapping ψ_{n+1} of V_{n+1}.

For the case with boundary a slight modification of the above construction and argument is needed.

Appendix: The Long Line. In order to show that second countability is necessary for the classification theorem we present here an example of a Hausdorff space which satisfies property (M) but is not second countable.

For this section some familiarity with transfinite ordinals is required. Consider the set L of all pairs (α, x) where α is a countable ordinal and $x \in [0, 1\rangle$ and order these pairs lexicographically, that is, $(\alpha, x) > (\beta, y)$ if $\alpha > \beta$ or if $\alpha = \beta$ and $x > y$. It is easy to see that L is a Hausdorff space in the order topology.

For each countable ordinal α define $H_\alpha = \{(\beta, x) \in L | \beta < \alpha\}$.

PROPOSITION 4. *H_α is an H-set.*

Granting Proposition 4, it follows at once that (M) is satisfied by the sets H_α, so L is a manifold with boundary. However, it is not second countable because the $\{H_\alpha\}$ cannot be reduced to a countable covering. Namely, if $\{H_{\alpha_i}\}$, $i = 1, 2, \ldots$, were such a covering, then any countable ordinal would be in some H_{α_i}, but there are only a countable number of ordinals in each H_{α_i}. Hence $\bigcup_{i=1}^{\infty} H_\alpha$ is a countable set, contradicting the fact that there are uncountably many countable ordinals.

To prove Proposition 4 one must show that (i) H_α has the least upper bound property, and (ii) H_α has a countable dense subset. Both are easily proved. Finally one uses the fact that any set with properties (i) and (ii) is order-isomorphic to a real interval, in this case a half-open interval. This is not hard either. First map the rational points of $[0, 1\rangle$ order-isomorphically onto the countable dense subset and then extend $[0, 1\rangle$ using the least upper bound property.

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The Error for Quadrature Methods: A Complex Variables Approach

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By a quadrature method for integration over an interval $[a, b]$, we mean a set of distinct points $x_0 < x_1 < \cdots < x_n$, a set of constants $\alpha_0, \ldots, \alpha_n$, and a formula

$$Q_n(f) := \sum_{j=0}^{n} \alpha_j f(x_j) \tag{1}$$

that serves as an estimate of the integral

$$\int_a^b f(x)\, dx.$$

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