A combinatorial model for totally nonnegative partial flag varieties

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Overview

- Grassmannians and their totally nonnegative parts Gr_{kn}^{tnn}
- Combinatorial description of Gr_{kn}^{tnn} in terms of networks
- Flag varieties and their totally nonnegative parts Fl^{tnn}
- Momentum-twistor diagrams (the combinatorial model for Fl^{tnn})

Grassmannians

The Grassmannian Gr_{kn} is the space of k-dimensional subspaces of $\mathbb{C}^n$.

$$X = \text{rowspan} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \hdotsfor{4} \\ x_{k1} & x_{k2} & \dots & x_{kn} \end{bmatrix}$$

Any full rank $k \times n$ matrix gives a point in Gr_{kn}.

Two matrices M and M' give the same point in Gr_{kn} if $M' = gM$ for some $g \in \mathrm{GL}_k$. So, $\mathrm{Gr}_{kn} = \mathrm{GL}_k \backslash \mathrm{Mat}_{kn}$.

For $I \in \binom{[n]}{k}$, the Plücker coordinate is

$$\Delta_I(X) = (k \times k) \text{ minor of } X \text{ using columns } I.$$

Example: The Plücker coordinates of the 2-plane

$$X = \text{rowspan} \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

are $\Delta_{12} = 1$, $\Delta_{13} = c$, $\Delta_{14} = d$, $\Delta_{23} = -a$, $\Delta_{24} = -b$, and $\Delta_{34} = ad - bc$.

They satisfy the Plücker relation $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.

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In general, a collection of numbers $\left((\Delta_I)_{I \in \binom{[n]}{k}}\right)$, not all zero, defines a point in Gr_{kn} if and only if the Plücker relation with $r = 1$ index swapped is satisfied:

$$\sum_{s=1}^{n} (-1)^s \Delta_{i_1, i_2, \ldots, i_{k-1}, j_s} \Delta_{j_1, \ldots, j_{s-1}, \hat{j}_s, j_{s+1}, \ldots, j_k} = 0$$

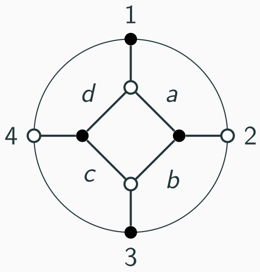
where $\hat{j}_s$ denotes omission.

The totally nonnegative Grassmannian Gr_{kn}^{tnn} is the set of elements in the Grassmannian Gr_{kn} with all nonnegative Plücker coordinates.

Combinatorial description of Gr_{kn}^{tnn} (by Postnikov)

A plabic graph G is a planar bipartite graph in a disk.

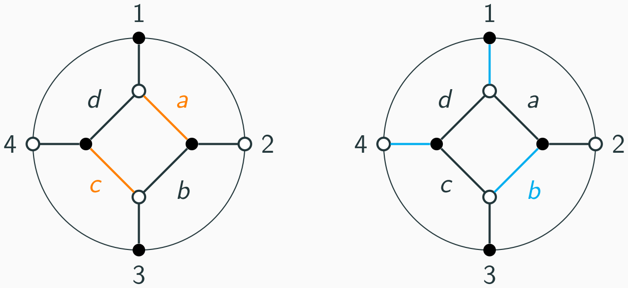
- vertices are either black or white, and every edge connects a black and a white vertex
- *n boundary vertices* on the boundary, labeled clockwise
- all boundary vertices have degree one, and there are no edges joining boundary vertices.



G with positive edge weights will be called a network.

There is a "boundary measurement" map $\mathrm{BM} : \{\text{Networks}\} \to \mathrm{Gr}_{kn}^{tnn}$

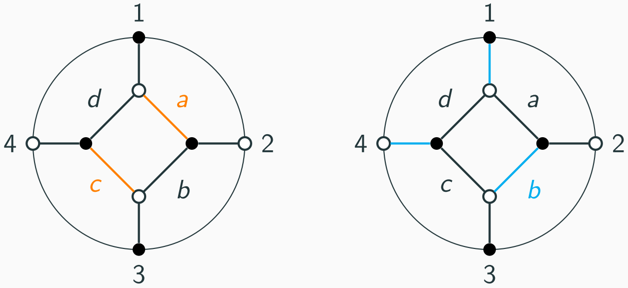
An almost perfect matching Π is a subset of edges of N such that
(1) each interior vertex is used exactly once,
(2) boundary vertices may or may not be used.



Almost perfect matchings Π_1 and Π_2.

The boundary subset $I(\Pi) \subset \{1, 2, \dots, n\}$

$I(\Pi)$ = black vertices used by Π and white vertices not used by Π.



Almost perfect matchings Π_1 and Π_2 such that $I(\Pi_1) = \{2, 4\}$ and $I(\Pi_2) = \{1, 2\}$.

For $I \in \binom{[n]}{k}$, the boundary measurement is

$$\Delta_I(N) = \sum_{I(\Pi)=I} \mathrm{wt}(\Pi)$$

$\mathrm{wt}(\Pi)$ is the product of the weights of the edges in Π.

Example: $\mathrm{Gr}(2,4)$ and $I = \{2,4\}$

For $I \in \binom{[n]}{k}$, the boundary measurement is

$$\Delta_I(N) = \sum_{I(\Pi)=I} \mathrm{wt}(\Pi)$$

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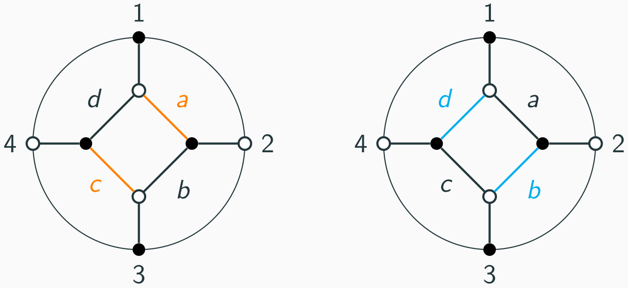
Example: $\mathrm{Gr}(2,4)$ and $I = \{2,4\}$



$$\Delta_{24}(N) = \mathrm{wt}(\Pi_1) + \mathrm{wt}(\Pi_2) = ac + bd$$

Boundary Measurements map BM

$$\begin{array}{ccc} \{\text{Networks}\} & \xrightarrow{\text{BM}} & \mathrm{Gr}_{kn}^{tnn} \\ N & \longmapsto & (\Delta_I(N))_{I \in \binom{[n]}{k}}. \end{array}$$

Theorem (Postnikov)

1) Well-definedness: given a network, its boundary measurements land in Gr_{kn}^{tnn} (i.e., they satisfy Plücker relations).
2) Surjectivity: every point in Gr_{kn}^{tnn} comes from some network.

In fact, something stronger can be said here.

Stratification of Gr_{kn}^{tnn}

Gr_{kn}^{tnn} can be stratified into positroid cells:
given by which Δ_I's are zero and which Δ_I's are positive.

These positroid cells

- are disjoint.
- closure of one is union of smaller ones.
- make up all of Gr_{kn}^{tnn}.

Let $\mathrm{BM}(G) = \{\mathrm{BM}(N) \mid N \text{ is a choice of edge weights for } G\}$.

Theorem (Postnikov)

3) Characterizing the image: $\{\mathrm{BM}(G)\} \overset{bij}{\longleftrightarrow} \{\textit{positroid cells}\}$.

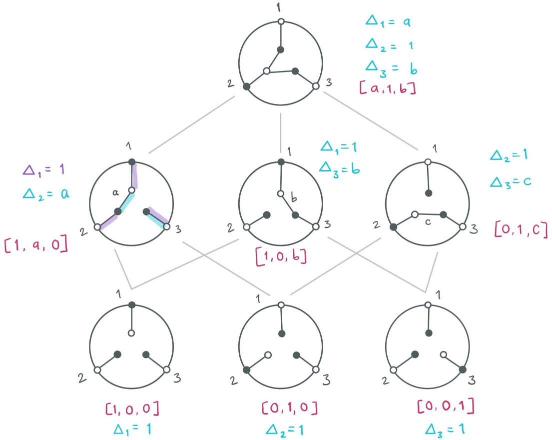
- stronger statement of surjectivity: "cell by cell" surjectivity.

$$\mathrm{BM}(G) = \{\mathrm{BM}(N) \mid N \text{ is a choice of edge weights for } G\}$$

Theorem (Postnikov)

4) Disjointness of images: for two graphs G and G', we have either $\mathrm{BM}(G) = \mathrm{BM}(G')$ or $\mathrm{BM}(G) \cap \mathrm{BM}(G') = \varnothing$.

The cell decomposition of Gr_{kn}^{tnn} is “induced” by the graphs.



Poset of Gr_{13}^{tnn}

Is there a combinatorial description for partial flag varieties?

Focus on two-step flag variety $\mathrm{Fl}(k, k+2; n)$.

We consider matrices of the form

$$X = \text{rowspan} \left[\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \hline x_{31} & x_{32} & \cdots & x_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{(k+2)1} & x_{(k+2)2} & \cdots & x_{(k+2)n} \end{array}\right]$$

The $\mathrm{Fl}(k, k+2; n)$. Two matrices $M, M' \in M(k+2, n)$ give the same point in $\mathrm{Fl}(k, k+2; n)$ if $M' = g.M$ for some $g \in GL(k; 1)$, where $GL(k; 1)$ is the group of all invertible $(k+2) \times (k+2)$ matrices of the form

$$g = \begin{bmatrix} A_{2\times 2} & \mu_{2\times k} \\ 0_{k\times 2} & B_{k\times k} \end{bmatrix}$$

The nonnegative part of $\mathrm{Fl}(k, k+2; n)$.

$$X = \text{rowspan} \left[\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \hline x_{31} & x_{32} & \cdots & x_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{(k+2)1} & x_{(k+2)2} & \cdots & x_{(k+2)n} \end{array}\right]$$

We say that a matrix $M \in \mathrm{Fl}(k, k+2; n)^{tnn}$ if $M \in \mathrm{Gr}^{tnn}_{(k+2)n}$ and $M_0 \in \mathrm{Gr}^{tnn}_{kn}$, where M_0 denotes the matrix M with the first two rows removed.

Both positivity conditions are invariant under GL(k;1) transformations.

Momentum-twistor diagram

A plabic graph is a planar bipartite graph in a disk.

- vertices are either black or white, and every edge connects a black and a white vertex.
- *n boundary vertices* on the boundary, labeled clockwise.
- all boundary vertices have degree one, and there are no edges joining boundary vertices.

Momentum-twistor diagram

A ~~plabic graph~~ momentum-twistor diagram is a planar bipartite graph ~~in a disk~~ on an annulus.

- vertices are either black or white, and every edge connects a black and a white vertex.
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Momentum-twistor diagram

A ~~plabic graph~~ momentum-twistor diagram is a planar bipartite graph ~~in a disk~~ on an annulus.

- vertices are either black or white, and every edge connects a black and a white vertex.
- *n boundary vertices* on the boundary, labeled clockwise.
- two *puncture vertices* on the inner boundary
- all boundary vertices have degree one, and there are no edges joining boundary vertices.

A momentum-twistor diagram M is a planar bipartite graph on an annulus.

- vertices are either black or white, and every edge connects a black and a white vertex.
- *n boundary vertices* on the boundary, labeled clockwise.
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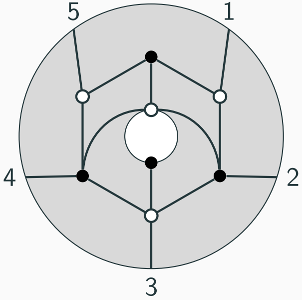
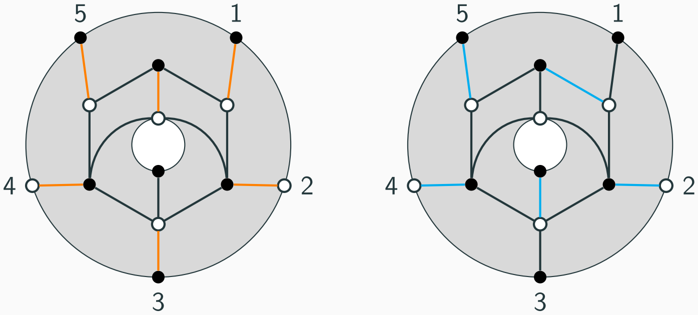


An almost perfect matching Π is a subset of edges of M such that

(1) each interior vertex is used exactly once,

(2) boundary vertices may or may not be used,

(3) exactly one of the puncture vertex is used.



Almost perfect matchings Π_1 and Π_2 such that $I(\Pi_1) = \{1, 3, 5\}$ and $I(\Pi_2) = \{5\}$.

For $J \in \binom{[n]}{k}$, the boundary measurement is

$$\Delta_J(M) = \sum_{I(\Pi)=J} \mathrm{wt}(\Pi)$$

$\mathrm{wt}(\Pi)$ is the product of the weights of the edges in Π.

Theorem (Fraser-K-Matherne)

The boundary meas map $\{$*MTD with non-neg edge wts*$\} \to \mathrm{Fl}^{tnn}$
is well defined.

is surjective.

Surjectivity proved for $\mathrm{Fl}(1,3;n)$ and $\mathrm{Fl}(2,4;n)$.
In progress for $\mathrm{Fl}(k,k+2;n)$.

	Gr_{kn}^{tnn}	$\mathrm{Fl}_{(k,k+2;n)}^{tnn}$
1.	BM is well defined	BM is well defined
2.	BM is surjective	BM is surjective proved for $(1,3;n)$ and $(2,4;n)$
3.	"cell by cell" surjectivity $\{\mathrm{BM}(G)\} \stackrel{\text{bij}}{\longleftrightarrow} \{\text{positroid cells}\}$	in examples quadratic rel in plucker coords
4.	disjointness of images: BM(G) = BM(G') or BM(G) $\cap$ BM(G') $= \emptyset$	not true here BM(G) $\neq$ BM(G') and BM(G) $\cap$ BM(G') $\neq \emptyset$

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Thank you!