Minisymposium 24

Probability and Geometry

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The goal of this special session is to bring together people working in Analysis, Geometry or Probability and to focus on recent developments in *Geometric Analysis on Riemannian manifolds, Diffusion processes on fractal and singular spaces, Stochastic differential geometry, Stochastic analysis in infinite dimensions*.

It is well-known that Brownian motion and martingales on manifolds or vector bundles connect local and global geometry in an intrinsic way, and that many questions related to the geometry of Laplace operators have a direct probabilistic counterpart. Furthermore probabilistic methods often extend naturally to areas, like singular spaces or infinite dimensional spaces, where standard tools of differential analysis or PDE methods fall short. This Mini-Symposium intends to focus on recent progress in these areas.

Donnerstag, 21. September

Übungsraum 3, Geographisches Institut, Meckenheimer Allee 166

14:00 – 14:45 **Thierry Coulhon** *(Cergy)*
Large time behavior of heat kernels on forms

14:45 – 15:30 **Shigeki Aida** *(Osaka)*
Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold

15:30 – 16:00 *Coffee break*

16:00 – 16:45 **Ana Bela Cruzeiro** *(Lisbon)*
On a stochastic Euler equation

16:45 – 17:30 **Feng-Yu Wang** *(Beijing)*
Estimates of the first Neumann eigenvalue and the log-Sobolev constant on non-convex manifolds

17:30 – 18:15 **Fuzhou Gong** *(Beijing)*
Exponential ergodicity, spectral gap, and their applications

Freitag, 22. September

Übungsraum 3, Geographisches Institut, Meckenheimer Allee 166

14:00 – 14:45 **Ichiro Shigekawa** *(Kyoto)*
One dimensional diffusions conditioned to be non-explosive

14:45 – 15:30 **Roland Friedrich** *(MPI Bonn)*
Diffusions on moduli spaces and generalised Stochastic Loewner Evolutions

15:30 – 16:00 *Coffee break*

16:00 – 16:45 **José A. Ramírez** *(Costa Rica)*
Beta ensembles, stochastic Airy spectrum, and a diffusion

16:45 – 17:30 **Peter K. Friz** *(Cambridge)*
Stochastic processes as rough paths and Carnot-Caratheodory geometry

17:30 – 18:15 **Marc Arnaudon** *(Poitiers)*
Gradient estimates for positive harmonic functions, Harnack inequalities and heat kernel estimates on Riemannian manifolds, by stochastic analysis

18:15 – 19:00 **Jinghai Shao** *(Beijing)*
Optimal transportation maps for Monge-Kantorovich problem on loop groups

Vortragsauszüge

Thierry Coulhon *(Cergy)*
Large time behavior of heat kernels on forms

This is a report on a joint work with Qi S. Zhang. We derive large time upper bounds for heat kernels on vector bundles of differential forms on a class of non-compact Riemannian manifolds under certain curvature conditions.

Shigeki Aida *(Osaka)*
Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold

We determine the limit of the bottom of spectrum of Schrödinger operators with variable coefficients on Wiener spaces and path spaces over finite dimensional compact Riemannian manifolds under semi-classical limit. The problem on path spaces over Riemannian manifolds are considered as a problem on Wiener spaces by Ito's map. However the coefficient operator is not a bounded linear operator and the dependence on the path is not continuous in the uniform convergence topology if the Riemannian curvature tensor on the underling manifold is not equal to 0. The difficulties are solved by using unitary transformations of the Schrödinger operators by approximate ground state functions and estimates in the rough path analysis.

Ana Bela Cruzeiro *(Lisbon)*
On a stochastic Euler equation

We follow Arnold's approach of Euler equation as a geodesic on the group of diffeomorphisms and prove the existence of a stochastic perturbation of this equation when the underlying manifold is the two dimensional torus.

This is joint work with F. Flandoli and P. Malliavin.

Feng-Yu Wang *(Beijing)*
Estimates of the first Neumann eigenvalue and the log-Sobolev constant on non-convex manifolds

A number of explicit lower bounds are presented for the first Neumann eigenvalue on non-convex manifolds. The main idea to derive these estimates is to make a conformal change of the metric such that the manifold is convex under the new metric, which enables one to apply known results obtained in the convex case. This method also works for more general functional inequalities. In particular, some explicit lower bounds are presented for the log-Sobolev constant on non-convex manifolds.

Fuzhou Gong *(Beijing)*
Exponential ergodicity, spectral gap, and their applications

In this talk, first we give some reason for ergodic theory from our knowledge. Secondly, we present a characterization of spectral gap for positive operators and positive C_0-semigroups in L^p-space with $1 < p < +\infty$, and we describe an equivalent relation between spectral gap and exponential ergodicity of Markov chains or Markov processes. As application, we give the existence of spectral gap to Donsker's invariance principle and Strassen's strong invariance principle for Markov chains or Markov processes, as well as some results on the existence of spectral gap for Schroedinger operators and Girsanov semigroups. Finally, we introduce background and mathematical framework of the mass gap (or spectral gap) problem on loop spaces; we give a survey on this problem and formulate some important open problems on loop spaces concerning this problem.

Ichiro Shigekawa *(Kyoto)*
One dimensional diffusions conditioned to be non-explosive

We consider one dimensional diffusions conditioned to be non-explosive. Suppose we are given a minimal diffusion process $\{X_t, P_x\}$ on an interval (l_1, l_2). Let ζ be its explosion time. If $P_x[\zeta = \infty] > 0$, then the measure conditioned to be non-explosive is defined by

$$P_x[\,\cdot\,|\,\zeta=\infty]=P_x[\,\cdot\,\cap\zeta=\infty]/P_x[\zeta=\infty].$$

If $P_x[\zeta=\infty]=0$, then the measure conditioned to be non-explosive is defined as the limit

$$\lim_{T\to\infty} P_x[\,\cdot\,|\,\zeta>T].$$

If the limit exists and the limit is a diffusion process, we call it a *surviving diffusion*. We are interested in the following problems:

(1) When does the surviving diffusion exist?
(2) Characterization of the surviving diffusion.

The surviving diffusion is characterized as a h-transform of the original process by the λ-harmonic function φ, λ being the principal eigenvalue.

Roland Friedrich *(MPI Bonn)*
Diffusions on moduli spaces and generalised Stochastic Loewner Evolutions

In this talk we shall discuss a very general construction principle of measures on paths on Riemann surfaces. These curves naturally arise e.g. as the fluctuating phase boundaries of statistical mechanics models in the scaling limit. The fundamental observation is that a certain class of diffusion processes on a dressed moduli space generates random paths / sets on the surfaces themselves.
In our framework we obtain the “ordinarySStochastic Lœwner Evolution (SLE), as a special case; thereby showing the underlying global geometric structure, as well.
Further, via the representation theory of infinite dimensional Lie algebras, we shall make contact with other mathematical / physical fields, in particular with Conformal Field Theory (CFT).

José A. Ramírez *(Costa Rica)*
Beta ensembles, stochastic Airy spectrum, and a diffusion

This talk will be about a connection between stochastic differential operators and the standard ensembles of Random Matrix Theory. It is joint work with B. Rider.
Building on earlier work of A. Edelman, I. Dumitriu, and B. Sutton we prove that the largest eigenvalues of the general beta-ensemble of Random Matrix Theory, properly

centered and scaled, converge in distribution to the law of the low lying eigenvalues of a random operator of Schrödinger type. The latter is

$$-\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

acting on $L^2(R_+)$ with Dirichlet boundary condition at $x = 0$. Here $b'(x)$ denotes a standard White Noise and the $\beta > 0$ is that of the original ensemble.
Based on this convergence, we provide a new characterization of the Tracy-Widom type laws (for all β) in terms of the explosion/non-explosion a one-dimensional diffusion.

Peter K. Friz *(Cambridge)*
Stochastic processes as rough paths and Carnot-Caratheodory geometry

Brownian motion on the step-n free nilpotent group with d generators is a well-known object; in particular, there are Gaussian heat-kernel bounds in term of the Carnot-Caratheodory metric. The resulting sample path regularity is exactly the required regularity in the sense of Lyons' rough path theory. In fact, it suffices to consider n=2, that is, standard Brownian motion and Levy's area.
If one replaces standard Brownian motion by (i) a continuous martingale or (ii) a suitable Gaussian process there are alternative ways to construct Levy's area. The required sample path regularity in the rough path sense can be shown via old ideas from Lepingle and Wiener-Ito chaos integrability respectively.
Finally, if one considers (iii) Markov process with uniformly sub-elliptic generators the theory of Dirichlet forms yields Gaussian heat-kernel bounds and we obtain a large class of rough paths. A support description on path space was conjectured by T. Lyons and we will report on some progress in this direction.

Joint work with N. Victoir.

Marc Arnaudon *(Poitiers)*
Gradient estimates for positive harmonic functions, Harnack inequalities and heat kernel estimates on Riemannian manifolds, by stochastic analysis

The talk is divided into three parts; we report on recent work with Bruce Driver, Anton Thalmaier and Feng-Yu Wang.

In the first part we prove gradient estimates for positive harmonic functions on Riemannian manifolds by using a Bismut type inequality which is derived by an integration by parts argument from an underlying submartingale. A crucial but elementary ingredient is that positive local martingales have moments of order $\beta \in]0,1[$ dominated by $C_\beta\, z$ where C_β is a universal positive constant and z is the starting point of the local martingale.
In the second part, coupling by parallel translation, along with Girsanov's theorem, is used to establish a new version of a dimension-free Harnack inequality for diffusion semigroups on Riemannian manifolds with Ricci curvature unbounded below. As an application, in the symmetric case, a Li-Yau type heat kernel bound is presented for such semigroups.
In the third part we prove Li-Yau and Hamilton estimates for heat kernels in compact manifolds by replacing classical maximum principle by submartingale arguments. For Hamilton's estimate, we demonstrate that a certain quadratic form valued semimartingale can not exit the set of nonpositive quadratic forms, outside of which it would have a drift contradicting its known asymptotic behaviour.

Jinghai Shao *(Beijing)*

Optimal transportation maps for Monge-Kantorovich problem on loop groups

Monge-Kantorovich problem is to consider how to move one distribution to another one as efficiently as possible. The efficiency is measured w.r.t. a cost function $c(x, y)$. It is naturally connected with the Wasserstein distance between two measures and also with the transportation cost inequality. In this work, we consider the Monge-Kantorovich problem on loop groups. Let G be a compact Lie group, and consider the loop group $\mathcal{L}_eG := \{\ell \in C([0,1], G);\ \ell(0) = \ell(1) = e\}$. Let ν be the heat kernel measure at time 1. For any density function F w.r.t. ν on $\mathcal{L}_eG$ with $\mathrm{Ent}_\nu(F) < \infty$, we shall show that there exists a unique optimal transportation map $\mathcal{T} : \mathcal{L}_eG \to \mathcal{L}_eG$ which pushes ν forward to $F\nu$. Our work is based partly on McCann's result on the Riemannian manifold (2001) and partly on the Feyel and Üstünel's work (2002), where they treated the Monge-Kantorovich problem in the abstract Wiener space.