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INTERVAL EXCHANGE SPACES AND MODULI SPACES

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ABSTRACT. Spaces of interval exchanges are introduced. In particular, a map from the moduli space of directed Riemann surfaces of genus g to the space of interval exchanges of rank at most $4g$ is studied.

0. Introduction.

To cut the real line at finitely many points and to reglue the pieces in another order is called an interval exchange. This type of non-continuous self-maps arises in various contexts, for example, as first-return-maps of flows on surfaces, or when playing billiard in a polygon. They are measure-preserving with interesting properties, and are therefore well-studied in ergodic theory. For a general introduction see the books [Sinai], [Cornfeld-Fomin-Sinai], [Mañe], and the articles [Arnoux-Ornstein-Weiss 1985], [Keane 1975, 1977], [Keane-Rauzy 1980], [Rauzy 1976/77, 1979], [Veech 1978, 1984]; for the connection to flows on surfaces see [Katok 1980], [Arnoux 1981, 1988], [Arnoux-Levitt 1986], [Arnoux-Yoccoz 1981], [Levitt 1980, 1982, 1983], [Kerckhoff 1985] and [Rees 1981].

Usually the dynamical behaviour of a single exchange under iteration is studied. In this article we want to emphasize that these exchanges form interesting spaces $\mathfrak{Ex}(n)$, where $n-1$ is the number of intervals exchanged. Our interest comes from studying the moduli space $\vec{\mathfrak{M}}(g)$ of directed Riemann surfaces, i.e. a closed Riemann surface together with a tangent direction at some basepoint given. Such a surface comes with a harmonic flow and a canonical curve, on which to induce a first-return map. To associate in this manner to a Riemann surface an exchange, defines a continuous map $\Phi : \vec{\mathfrak{M}}(g) \to \mathfrak{Ex}(4g)$. The idea of such a close connection between the moduli of Riemann surfaces and interval exchanges occurs at several places, e.g. [Masur 1982], [Veech 1982] and [Strebel 1984]; in [Bödigheimer 1990 II] interval exchanges were used as a technical mean to construct certain operations. Here we will treat Φ as a globally defined, continuous map.

Moduli spaces and interval exchange spaces have several structures in common, all preserved by Φ. This is somewhat surprising, since the moduli spaces

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are based on two-dimensional geometry and analysis, whereas interval exchanges are a one-dimensional, almost completely combinatorial phenomenon.

After determining their homotopy type, it becomes obvious that the spaces $\mathfrak{Ex}(n)$ need to be refined in order to make them more useful for the investigation of moduli spaces. Nevertheless, it seems worthwhile to introduce them as a first step.

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1. Interval exchange spaces.

As explained above an interval exchange (on the real line) is a reglueing of subintervals in a permuted order. Thus it is determined by an increasing sequence $y_0 \leq y_1 \leq \cdots \leq y_{n-1}$ of (cut) points in $\mathbb{R}$ together with a permutation π in the $(n-1)$th symmetric group $\mathfrak{S}_{n-1}$ (acting on the indices $0, 1, \ldots, n-2$ as we see later). We always have $n \geq 2$, and our notation is $y = (y_0, \ldots, y_{n-1}|\pi)$.

Note we are not assuming the points y_i to be distinct; but we do exclude the totally degenerate case $y_0 = y_1 = \cdots = y_{n-1}$ by assuming that $y_0 < y_{n-1}$. The length of the subinterval $Y_i = [y_i, y_{i+1}]$ is denoted by t_i, and $t = (t_0, \ldots, t_{n-2})$ is called the length vector, and $\|t\| = \Sigma t_i > 0$ is called the length of y. We say y is of rank less or equal to $n-1$. If $y_0 = 0$ (resp. $y_0 = 0$ and $y_{n-1} = 1$) we call y half-normalized (resp. normalized).

So far we have emphasized the combinatorial point of view; for the dynamical point of view we regard the cutting, permuting and reglueing as a discontinuous self-map of $\mathbb{R}$, with compact support and finitely many points of discontinuity, which is isometric and orientation-preserving on the subintervals of continuity. To make this precise, set $\tilde{y}_0 = y_0$ and $\tilde{y}_{i+1} = \tilde{y}_i + t_{\pi(i)}$ for $i = 0, \ldots, n-1$; thus $y_0 = \tilde{y}_0 \leq \tilde{y}_n \leq \cdots \leq \tilde{y}_{n-1} = y_{n-1}$. The subinterval $\tilde{Y}_i = [\tilde{y}_i, \tilde{y}_{i+1}]$ has the same length as $Y_{\pi(i)}$, and altogether they cover the same interval $[y_0, y_{n-1}]$, which we call the support of y. The self-map $f_y : \mathbb{R} \to \mathbb{R}$ is now almost determined by saying that f_y maps the interior of $\tilde{Y}_i$ onto the interior of $Y_{\pi(i)}$ isometrically and orientation-preserving, and that f_y is the identity outside of $[y_0, y_{n-1}]$; or in formulas

$$(1.1) \qquad f_y(\xi) = y_{\pi(i)} + (\xi - \tilde{y}_i) \quad \text{for} \quad \tilde{y}_i < \xi < \tilde{y}_{i+1} \,.$$

We extend the definition to the cut points $\tilde{y}_0, \ldots, \tilde{y}_{n-1}$, by requiring f_y to be continuous from the right. We are actually only interested in the L^2-class $[f_y]$ of f_y, i.e. we identify two such self-maps if they agree up to finitely many points.

This suggests that we introduce the corresponding equivalence relation for $y = (y_0, \ldots, y_{n+1}|\pi)$ by ignoring the subintervals of length zero. The easiest way to express this is by rewriting $y = (y_0, y_1, \ldots, y_{n-1}|\pi)$ as $y = (y_0; t_0, \ldots, t_{n-2}|\pi)$. Then the equivalence relation is generated by the following.

$$(1.2) \quad (y_0; t_0, \ldots, t_i, t_{i+1}, \ldots, t_{n-2}|\pi) \sim (y_0; t_0, \ldots, t_{i+1}, t_i, \ldots, t_{n-2}|\langle i\ i{+}1\rangle \circ \pi)$$

if $t_i = 0$, and

(1.3) $(y_0; t_0, \dots, t_i, t_{i+1}, \dots, t_{n-2}|\pi) \sim (y_0; t_0, \dots, t_i, t_{i+1}, \dots, t_{n-2}|\pi\circ\langle j\ j{+}1\rangle)$

if $t_i = 0$ and $\pi(j) = i$. Here $\langle k\ k+1\rangle \in \mathfrak{S}_{n-1}$ is the transposition of k and $k+1$, for $0 \leq k < n-2$. (1.2) allows subintervals of length zero to be moved among the Y_i, and (1.3) among the $\tilde{Y}_i$. But this equivalence relation does not remove unnecessary cut points $\tilde{y}_i$ (where f_y is continuous). The equivalence class of y is denoted by $\mathfrak{y} = [y] = [y_0, y_1, \dots |\pi] = [y_0; t_0, \dots |\pi]$.

Let $\mathfrak{Ex}(n)$ (resp. $\mathfrak{Ex}'(n)$, resp. $\mathfrak{Ex}''(n)$) denote the set of equivalence classes of normalized (resp. half-normalized, resp. all) interval exchanges. To put a topology on $\mathfrak{Ex}(n)$ it is natural to regard the t_i as barycentric coordinates in an $(n-2)$-simplex Δ^{n-2} indexed by the permutation π — at least as long as all t_i are positive; and the equivalence relation tells us how to identify a face of a simplex with other faces of the same and other simplices. Thus $\mathfrak{Ex}(n)$ is a finite cell complex,

(1.4) $$\mathfrak{Ex}(n) = (\Delta^{n-2} \times \mathfrak{S}_{n-1})/\sim$$

For $\mathfrak{Ex}'(n)$ resp. $\mathfrak{Ex}''(n)$ the topology is the obvious one, using the relation $\mathfrak{Ex}'(n) \cong \mathbb{R}_+ \times \mathfrak{Ex}(n)$, resp. $\mathfrak{Ex}''(n) \cong \mathbb{R}_+ \times \mathbb{R} \times \mathfrak{Ex}(n)$, where the parameter in the positive reals $\mathbb{R}_+$ is the length $\|t\|$, and the parameter in $\mathbb{R}$ is y_0.

As we will find this topology on is rather crude, but makes it easy to find the homotopy type of $\mathfrak{Ex}(n)$.

2. Some properties of the spaces $\mathfrak{Ex}(n)$.

The spaces $\mathfrak{Ex}(n)$ have various interesting structural properties almost all of which correspond to properties of moduli spaces, as we shall see in the next section.

The normalizations we made come from the action of the group $\mathrm{Sim}(\mathbb{R})$ of similarities of the real line, generated by translations and dilations. $\mathrm{Sim}(\mathbb{R})$ is the semi-direct product $\mathbb{R} \rtimes \mathbb{R}_+$ of the normal subgroup $\mathbb{R}$ of translations $\xi \longmapsto \xi+b$, $b \in \mathbb{R}$, and the quotient group $\mathbb{R}_+$ of dilations $\xi \longmapsto a\xi$, $a > 0$. The action on an interval exchange y is $y+b = [y_0+b, \dots, y_{n-1}+b|\pi]$ and $y.a = [ay_0, \dots, ay_{n-1}|\pi]$. The action is free and comes with a section $y \longmapsto (y_0, \|t\|) \in \mathrm{Sim}(\mathbb{R})$.

There are various ways to map $\mathfrak{Ex}(n)$ to $\mathfrak{Ex}(n+1)$. The first, called inclusion, is given by

(2.1) $$\iota : \mathfrak{Ex}(n) \to \mathfrak{Ex}(n+1),\ \iota[y_0, \dots, y_{n-1}|\pi] = [y_0, \dots, y_{n-1}, y_{n-1}|\pi \oplus id]\ ,$$

where $\pi \oplus id$ denotes the permutation in $\mathfrak{S}_n$ acting as π on the first $n-1$ indices and fixing the last one. The formula is also valid for spaces $\mathfrak{Ex}'(n)$ and $\mathfrak{Ex}''(n)$. If we take $\mathbb{I}_n = [0, 1, \dots, 1|\ \mathrm{id}]$ to be the basepoint in $\mathfrak{Ex}(n)$, then $\iota(\mathbb{I}_n) = \mathbb{I}_{n+1}$.

A second map is called stabilization and is defined by

(2.2) $$\begin{aligned}&\mathrm{Stab}\ : \mathfrak{Ex}(n) \to \mathfrak{Ex}(n+1)\ ,\\ &\mathrm{Stab}\ ([y_0, \dots, y_{n-1}|\pi]) = [\frac{y_0}{2}, \dots, \frac{y_{n-1}}{2}, 1|\pi \oplus id]\ .\end{aligned}$$

It is homotopic to ι.

Next we will find two ways to combine two exchanges to obtain a new one, i.e. maps $\mathfrak{Ex}(n) \times \mathfrak{Ex}(m) \to \mathfrak{Ex}(n+m)$. The Whitney sum emphasizes the combinatorial aspect; it is defined by putting the two exchanges side by side:

$$\begin{aligned} &\mathrm{Wh}([y_0,\dots,y_{n-1}|\pi],[y'_0,\dots,y'_{m-1}|\pi']) = \\ (2.3)\qquad &\qquad [\frac{y_0}{2},\dots,\frac{y_{n-1}}{2},\frac{1+y'_0}{2},\dots,\frac{1+y'_{m-1}}{2}|\pi\oplus\pi'] \,. \end{aligned}$$

The basepoints $\mathbb{I}_n$ behave like homotopy-units, because $\mathrm{Wh}(\mathfrak{y},\mathbb{I}_m) = \mathrm{Stab}^m(\mathfrak{y})$. Wh is obviously homotopy-associative. We shall see later that it is even homotopy-commutative. Thus $\mathfrak{Ex} = \coprod_{n\geq 2}\mathfrak{Ex}(n)$ is an H-space.

Using the fact that an exchange $\mathfrak{y}$ is almost the same thing as the map $[f_{\mathfrak{y}}]$, we define the composition (product) for unnormalized exchanges

$$(2.4)\qquad \mathrm{Comp}([y_0,\dots,y_{n-1}|\pi],[y'_0,\dots,y'_{m-1}|\pi']) = [y''_0,\dots,y''_{n+m-1}|\pi'']$$

where the new cut points and the new permutation are obtained as follows. Let $\tilde{y}_0 \leq \dots \leq \tilde{y}_{n-1}$ be the points of discontinuity of y. The cut points $y'_0 \leq \dots \leq y'_{m-1}$ of y' fall into groups according to which of the intervals $[\tilde{y}_j,\tilde{y}_{j+1}[$ contains them $(j=0,\dots,n-2)$. We rename them as

$$(2.5)\qquad y'_{j1} \leq y'_{j2} \leq \dots \leq y'_{ja_j} \in [\tilde{y}_j,\tilde{y}_{j+1}[$$

with $a_j \geq 0$, and $\sum_{j=0}^{n-2} a_j = m$. The function $f = f_y$ will map the interval $[\tilde{y}_j,\tilde{y}_{j+1}[$ to $[y_i,y_{i+1}[$ if $i=\pi(j)$. Setting

$$\begin{aligned} &y_{i0} := y_i && (i=0,\dots,n-1) \\ (2.6)\qquad & y_{i1} := f(y'_{j1}),\dots,y_{ia_j} := f(y'_{ja_j}) && (i=0,\dots,n-2) \end{aligned}$$

gives the new sequence of $n+m-1$ cut points y''_k:

$$(2.7)\quad y_{00} \leq y_{01} \leq \dots \leq y_{0b_0} \leq y_{10} \leq \cdots \qquad y_{n-2\,0} \leq \dots \leq y_{n-2\,b_{n-2}} \leq y_{n-1\,0}$$

where $b_i = a_{\pi^{-1}(i)}$. The new permutation will be a shuffle of π' into π, defined as follows. The cutpoints y'_j subdivided the interval $[\tilde{y}_j,\tilde{y}_{j+1}[$ into a_j+1 subintervals. Vice versa, the points $\tilde{y}_j$ subdivide the intervals $[y'_k,y'_{k+1}[$ into, say c_k+1 subintervals, with $c_k \geq 0$, $n = \sum_{k=0}^{m-2} c_k$. Pulling this subdivision back with $f' = f_{y'}$ gives a subdivision of $[\tilde{y}'_\ell,\tilde{y}'_{\ell+1}[$, if $k=\pi'(\ell)$. Set $d_\ell = c_{\pi'(\ell)}$. Now let $0 \leq r \leq n+m-2$ be given, and write

$$(2.8)\qquad r = d_0 + \dots + d_{s-1} + (s-1) + r_s \quad , \quad 0 < r_s \leq d_s + 1 \,.$$

Applying f' to the r-th interval of the pulled back subdivision gives us the r'-th interval of the (middle) subdivision, and

$$r' = d_{\pi'(0)} + \cdots + d_{\pi'(s-1)} + (s-1) + r_s \ . \tag{2.9}$$

Write r' as

$$r' = a_0 + \cdots + a_{t-1} + (t-1) + r'_t \quad , \quad 0 < r'_t \le a_t + 1 \ , \tag{2.10}$$

and the number of the image interval under $f = f_y$ will be

$$\pi(k) := a_{\pi(0)} + \cdots + a_{\pi(t-1)} + (t-1) + r'_t \ . \tag{2.11}$$

The product Comp is strictly-associative, and for the units we find $\mathrm{Comp}(\mathfrak{y}, \mathbb{I}_m) = \iota^m(\mathfrak{y}) = \mathrm{Comp}(\mathbb{I}_m, \mathfrak{y})$. Also later we will see that Comp is homotopy-commutative; indeed, we will show that Wh and Comp are homotopic, despite their apparent difference as to the dynamical behaviour; Wh always creates a decomposable system. Wh can be expressed as $\mathrm{Wh}(\mathfrak{y}, \mathfrak{y}') = \mathrm{Comp}(\frac{\mathfrak{y}}{2}, \frac{1+\mathfrak{y}'}{2})$; i.e. Wh is a special case of Comp, when the supports are disjoint. For the functions we have $[f_{\mathfrak{y}''}] = [f_{\mathfrak{y}}] \circ [f_{\mathfrak{y}'}]$.

Even more interesting are certain operations modelled after the Dyer-Lashof-operations in homotopy-theory. The "multiplications" Wh and Comp depend on several choices of parameters, and it is a non-trivial fact that there is an S^1-family of such multiplications, in which they are just two. This family is a map

$$\tilde{\theta} : S^1 \times \mathfrak{Ex}(n) \times \mathfrak{Ex}(m) \to \mathfrak{Ex}(n+m) \tag{2.12}$$

which is $\mathbb{Z}_2$-equivariant in the sense that it descends to a map

$$\theta : S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathfrak{Ex}(n)^2 \to \mathfrak{Ex}(2n) \tag{2.13}$$

where $\mathbb{Z}_2$ acts on S^1 antipodally and on $\mathfrak{Ex}(n)^2$ by switching factors. To define $\tilde{\theta}$ we need eight consecutive homotopies, all of the form $\mathrm{Comp}(\mathfrak{y}(\alpha), \mathfrak{y}'(\alpha))$ for $0 \le \alpha \le 8$, where $\mathfrak{y}(\alpha)$ and $\mathfrak{y}'(\alpha)$ are two paths in $\mathfrak{Ex}''(n)$ resp. $\mathfrak{Ex}''(m)$ starting and ending at $\mathfrak{y}(0) = \frac{\mathfrak{y}}{2}$, resp. $\mathfrak{y}'(0) = \frac{\mathfrak{y}+1}{2}$. To make formulas easier, given $v^- < v^+$, let $M(\mathfrak{y}, v^-, v^+) \in \mathfrak{Ex}''(n)$ be that translation and dilation of $\mathfrak{y}$ which has support $[v^-, v^+]$.

We now give the first four homotopies. The first and the fourth are actually stationary; this is to make $\tilde{\theta}$ compatible with the corresponding structure on the moduli space, see (3.12).

$$(2.14) \qquad \begin{aligned} \mathfrak{y}(\alpha) &= M(\mathfrak{y}, 0, \tfrac{1}{2}) = \tfrac{\mathfrak{y}}{2} \\ \mathfrak{y}'(\alpha) &= M(\mathfrak{y}', \tfrac{1}{2}, 1) = \tfrac{\mathfrak{y}+1}{2} \qquad 0 \le \alpha \le 1 \ ; \end{aligned}$$

$$(2.15) \qquad \begin{aligned} \mathfrak{y}(\alpha) &= M(\mathfrak{y}(1), 0, \tfrac{\alpha}{2}) \\ \mathfrak{y}'(\alpha) &= M(\mathfrak{y}'(1), \tfrac{2-\alpha}{2}, 1) \qquad 1 \le \alpha \le 2 \ ; \end{aligned}$$

$$(2.16)\qquad \begin{aligned} \mathfrak{y}(\alpha) &= M(\mathfrak{y}(2), \tfrac{\alpha-2}{2}, 1) \\ \mathfrak{y}'(\alpha) &= M(\mathfrak{y}'(2), 0, \tfrac{4-\alpha}{2}) \qquad 2 \leq \alpha \leq 3\,; \end{aligned}$$

$$(2.17)\qquad \begin{aligned} \mathfrak{y}(\alpha) &= M(\mathfrak{y}(3), \tfrac{1}{2}, 1) = \tfrac{\mathfrak{y}+1}{2} \\ \mathfrak{y}'(\alpha) &= M(\mathfrak{y}'(3), 0, \tfrac{1}{2}) = \tfrac{\mathfrak{y}'+1}{2} \qquad 3 \leq \alpha \leq 4\,; \end{aligned}$$

The homotopies five to eight are defined inexactly the same way as the first four, but the rôles of $\mathfrak{y}$ and $\mathfrak{y}'$ interchanged. The actual formula for $\tilde{\theta}$ is for now

$$(2.18)\qquad \begin{aligned} \tilde{\theta}(\alpha, \mathfrak{y}, \mathfrak{y}') &= \mathrm{Comp}(\mathfrak{y}'(8\alpha), \mathfrak{y}(8\alpha)) \qquad \text{for} \quad 0 \leq \alpha \leq \tfrac{1}{2}\,, \\ &= \mathrm{Comp}(\mathfrak{y}(8\alpha), \mathfrak{y}'(8\alpha)) \qquad \text{for} \quad \tfrac{1}{2} \leq \alpha \leq 1\,. \end{aligned}$$

We have $\tilde{\theta}(0, \mathfrak{y}, \mathfrak{y}') = \mathrm{Wh}(\mathfrak{y}, \mathfrak{y}')$, $\tilde{\theta}(\frac{1}{4}, \mathfrak{y}, \mathfrak{y}') = \mathrm{Comp}(\mathfrak{y}', \mathfrak{y})$, $\tilde{\theta}(\frac{1}{2}, \mathfrak{y}, \mathfrak{y}') = \mathrm{Wh}(\mathfrak{y}', \mathfrak{y})$, $\tilde{\theta}(\frac{3}{4}, \mathfrak{y}, \mathfrak{y}') = \mathrm{Comp}\ (\mathfrak{y}, \mathfrak{y}')$. This shows that Wh and Comp are homotopic, and – at the same time – that both are homotopy-commutative.

There are two interesting involutions on $\mathfrak{Cx}(n)$. One, called the reverse, is in the notation of section 1 given by

$$(2.19)\qquad \mathrm{Rev} : \mathfrak{Cx}(n) \longrightarrow \mathfrak{Cx}(n),\ \ \mathrm{Rev}[y_0, \dots, y_{n-1}|\pi] = [\tilde{y}_0, \dots, \tilde{y}_{n-1}|\pi^{-1}]\,.$$

The conjugation is obtained by reflecting the real line at $\frac{1}{2}$,

$$(2.20)\qquad \mathrm{Conj} : \mathfrak{Cx}(n) \longrightarrow \mathfrak{Cx}(n),\ \ \mathrm{Conj}(\mathfrak{y}) = [y_0; t_{n-2}, \dots, t_0|\omega \circ \pi \circ \omega^{-1}]$$

when $\mathfrak{y} = [y_0; t_0, \dots, t_{n-2}|\pi]$ and $\omega \in \mathfrak{S}_{n-1}$ is the reflection $\omega(k) = n - 1 - k$.

3. Parallel slit domains.

We recall here a particular description of the moduli space of directed Riemann surfaces; for more details and proofs we refer the reader to [Bödigheimer 1990 I].

Let F be a compact Riemann surface of genus $g \geq 0$ without boundary. In addition to the conformal structure a point $p \in F$ and a tangential direction $\mathfrak{x}$ at this point are given. Here a tangential direction $\mathfrak{x} = (x)$ is a non-zero tangent vector x, up to a positive multiple. We denote the moduli space of such triples $[F, p, \mathfrak{x}]$ by $\vec{\mathfrak{M}}(g)$; here a conformal equivalence is a conformal homeomorphism $f : F \longrightarrow F'$ such that $f(p) = p'$ and $df(\mathfrak{x}) = \mathfrak{x}'$.

This moduli space projects onto the classical moduli space of closed surfaces. It is an orientable, non-compact manifold of dimension by $6g - 3$. It is homotopy-equivalent to the classifying space of the mapping class group $\vec{\Gamma}(g) = \pi_0\ \mathrm{Diff}^+(F, p, \mathfrak{x})$ of isotopy classes of orientation-preserving diffeomorphisms of F fixing p and $\mathfrak{x}$. This mapping class group is isomorphic to the mapping class group of a genus g surface with one boundary curve (to be fixed pointwise); it is an extension of the classical mapping class group of a closed surface by the fundamental group of its unit tangent bundle.

Using the methods of geometric function theory $\vec{\mathfrak{M}}(g)$ can be described as the configuration space of parallel slit domains. Let $L_0, L_1, \dots, L_{4g-1}$ be a sequence of horizontal slits in $\mathbb{C}$, each of which is unbounded to the left. Such a L_i is given by its right endpoint $z_i = (x_i, y_i)$. These slits are paired to each other by an involution $\lambda \in \mathfrak{S}_{4g}$. We call $L = (L_0, L_1, \dots, L_{4g-1}|\lambda)$ a configuration if the following conditions are satisfied :

$$(3.1) \qquad \lambda(i) \neq i \qquad \text{for} \quad i = 0, \dots, 4g-1\,,$$

$$(3.2) \qquad \lambda^2(i) = i \qquad \text{for} \quad i = 0, \dots, 4g-1\,,$$

$$(3.3) \qquad y_0 \leq y_1 \leq \cdots \leq y_{4g-1}\,,$$

$$(3.4) \qquad x_i = x_{\lambda(i)} \qquad \text{for} \quad i = 0, \dots, 4g-1\,.$$

Note that these conditions so far do not exclude slits to be equal or overlapping; indeed all of them could still be equal.

The basic idea is that a configuration L represents a surface $F(L)$ obtained as follows: for each pair of slits L_k and $L_{\lambda(k)}$ glue the upper bank of L_k to the lower bank of $L_{\lambda(k)}$, and vice versa; take p to be the point at infinity of $\mathbb{C}$, and take $\mathfrak{x}$ to be the direction of dx under the local chart $\zeta \longmapsto \frac{1}{\zeta}$. The complement of the slits in $\mathbb{C}$ determines the conformal structure.

The crucial points are, of course, to exclude those configurations which will lead to singular surfaces, and to find those configurations which lead to conformally equivalent surfaces. For the first point we introduce two more conditions on a configuration:

(3.5) The permutation $\sigma \in \mathfrak{S}_{4g-1}$ defined by $\sigma(k) := \lambda(k+1) \mod 4g$ $(k = 0, 1, \dots, 4g-1)$ has only one cycle.

(3.6) L does not contain a subconfiguration $\dots L_k, L_{k+1}, L_{k+2} \dots$ such that $\lambda(k) = k+2$, $y_k = y_{k+1} = y_{k+2}$, and $x_{k+1} \leq x_k = x_{k+2}$.

The condition (3.5) guarantees that $p = \infty$ is not a singular point. To see this, consider the smallest closed rectangle $S \subset \mathbb{C}$ with sides parallel to the x– or y–axis and containing all points z_k; we call it the support of L. Suppose left vertical side Y of S lies on the vertical $x = u$, then it is cut into intervals Y_i by the points (u, y_i), just as defined in section 1. By glueing the banks of slits as described above the new sequence of the vertical intervals $Y_0, Y_1, \dots, Y_{4g-1}$, will be such that Y_k is followed by $Y_{\sigma(k)}$; it is clear that the boundary of S is connected in $F(L)$ if and only if σ has but one cycle. And since the complement of S is a neighbourhood of $p = \infty$ in $F(L)$, the connectivity of its boundary curve means that p is a smooth point.

Likewise, the condition (3.6) guarantees that in $F(L)$ the point $(x_k, y_k) = (x_{k+2}, y_{k+2})$ is smooth; a violation of (3.6) means that the closed curve in $F(L)$ from $z_k = (x_k, y_k)$ to $p = \infty$ and back to $z_{k+2} = (x_{k+2}, y_{k+2})$ — which is nontrivial in the generic case of all slits being disjoint — has become null-homotopic.

The second crucial point requires consideration of the equivalence relation generated by the following type of jump.

(3.7) Assume $L_{k-1} \subseteq L_k$ (i.e. $x_{k-1} \leq x_k$ and $y_{k-1} = y_k$) and $k < \lambda(k)$; then L is equivalent to $L' = (L'_0, \dots, L'_{4g-1}|\lambda')$ where:

$$L'_i = L_i \quad \text{for } i = 0, \dots, k-2,$$

$$L'_i = L_{i+1} \quad \text{for } i = k-1, \dots, \lambda(k)-1,$$

$$L'_i \text{ has endpoint } (x_{k-1}, y_{\lambda(k)}) \quad \text{for } i = \lambda(k)$$

$$L'_i = L_i \quad \text{for } i = \lambda(k)+1, \dots, 4g-1,$$

$$\text{and} \quad \lambda' = \langle \lambda(k)\ \lambda(k)-1 \dots k\ k-1 \rangle \circ \lambda \circ \langle k-1\ k \dots \lambda(k)-1\ \lambda(k) \rangle\,.$$

In other words, L_{k-1} "jumps" over the longer pair L_k, $L_{\lambda(k)}$, taking a new position, whereas all other slits change at most their index (by a partial rotation $\langle \lambda(k) \dots k\ k-1 \rangle$ of the indices from $k-1$ to $\lambda(k)$). Clearly, $F(L)$ does only depend on the equivalence class of L.

An equivalence class will be denoted by $\mathfrak{L} = [L] = [L_0, \dots, L_{4g-1}|\lambda]$. All the conditions (3.1-6) are invariant under jumps except for 3.6; we therefore say $\mathfrak{L}$ is non-degenerate if all its representatives satisfy (3.1) to (3.6), and call it a parallel slit domain. The set of parallel slit domains is denoted by $PS\mathbb{C}(g)$; it is a quotient space of the space of non-degenerate configurations, which is naturally a subspace of $\mathbb{C}^{4g} \times \mathfrak{S}_{4g}$.

In a parallel slit domain $\mathfrak{L}$ slits can still overlap to some extent, but it is no longer possible that they all coincide; indeed, they can not lie on the same y-level, and therefore we have always $y_0 < y_{4g-1}$.

As to normalizations, we find the 3-dimensional contractible group $\mathrm{Sim}(\mathbb{C}) = \mathbb{R}^2 \rtimes \mathbb{R}_+$ of similarities of $\mathbb{C}$ still acting freely on $PS\mathbb{C}(g)$ by translations and dilations. The translations in x- resp. y-direction correspond to the undetermined integration constants of two harmonic functions, as we will see in section 4; and the dilations correspond to the undetermined length of the tangent vector x representing the direction $\mathfrak{x} = (x)$. This action, too, has a section, namely $\mathfrak{L} \longmapsto (\min(x_i), y_0, y_{4g-1} - y_0) \in \mathrm{Sim}(\mathbb{C})$.

We call $\mathfrak{L}$ normalized (resp. half-normalized) if $y_0 = 0$, $y_{4g-1} = 1$ and $\min(x_i) = 0$ (resp. if $y_0 = 0$ and $\min(x_i) = 0$). $\mathfrak{P}(g)$ resp. $\mathfrak{P}'(g)$ denotes the subspace of normalized resp. half-normalized parallel slit domains.

The main result in [Bödigheimer 1990 I] now asserts that the space $\mathfrak{P}(g)$ of normalized parallel slit domains is homeomorphic to the moduli space $\vec{\mathfrak{M}}(g)$ of directed Riemann surfaces; see [ibid., p. 149].

We recall now some structural properties of directed moduli spaces as described in [Bödigheimer 1990 II].

There is a stabilization Stab : $\mathfrak{P}(g) \to \mathfrak{P}(g+1)$, which glues a new handle, i.e. a torus with one boundary curve, into a surface $F(L)$ near its basepoint p. Since the definition in [Bödigheimer 1990 II] is slightly different from what we have in mind here, we state it explicitly:

$$\begin{aligned}&\mathrm{Stab}\,[(x_0,y_0),\dots,(x_{4g-1},y_{4g-1})|\lambda] = \\ (3.8)\qquad &[(x_0,\frac{y_0}{2}),\dots,(x_{4g-1},\frac{y_{4g-1}}{2}),(1,\frac{1}{2}),(0,1),(1,1),(0,1)|\lambda\oplus\lambda_1]\,.\end{aligned}$$

We wrote end points instead of slits. $\lambda_1 \in \mathfrak{S}_4$ is the permutation $\langle 0\ 2\rangle\langle 1\ 3\rangle$, acting in on the last four indices $4g, 4g+1, 4g+2$ and $4g+3$. We remark thatλ_1 is the only permutation in $\mathfrak{S}_4$ satisfying all three conditions (3.1, 2, 5).

The Whitney sum Wh: $\mathfrak{P}(g)\times\mathfrak{P}(g')\to\mathfrak{P}(g+g')$ is essentially the same as what was called sum operation in [Bödigheimer 1990 II, p. 2]. Its normalized definition is

(3.9)

$$\begin{aligned}&\mathrm{Wh}([(x_0,y_0),\dots,(x_{4g-1},y_{4g-1})|\lambda],[(x'_0,y'_0),\dots,(x'_{4g'-1},y'_{4g'-1})|\lambda'])\\ &=[(x_0,\frac{y_0}{2}),\dots,(x_{4g-1},\frac{y_{4g-1}}{2}),(x'_0,\frac{1+y'_0}{2},\dots,(x'_{4g'-1},\frac{1+y'_{4g'-1}}{2})|\lambda\oplus\lambda']\,.\end{aligned}$$

The composition

$$\text{(3.10)}\qquad \mathrm{Comp}:\mathfrak{P}(g)\times\mathfrak{P}(g')\to\mathfrak{P}(g+g')$$

was not given separately in [Bödigheimer 1990 II]. Its formula is cumbersome; for the y-coordinates it is the same formula as in (2.4); the x-coordinates x_i of the first configuration stay the same, but the x-coordinates x'_i of the second configuration must be translated to the right by the amount $b=\max(x_i)$, the maximal real part of any slit endpoint of the first configuration.

Wh is homotopy-associative, and Comp is strictly-associative. If we set $\mathbb{I}_1 = [(1,0),(0,1),(1,1),(0,1)|\langle 0\ 2\rangle\langle 1\ 3\rangle] \in \mathfrak{P}(1)$ as the standard elliptic curve and basepoint, then $\mathbb{I}_g = \mathrm{Comp}(\mathbb{I}_{g-1},\mathbb{I}_1)\in\mathfrak{P}(g)$ will be our basepoint. They behave like homotopy-units for Wh, and like units for Comp. The stabilization becomes $\mathrm{Stab}(\mathfrak{L}) = \mathrm{Wh}(\mathfrak{L},\mathbb{I}_1)$. Now we can define another an inclusion by

$$\text{(3.11)}\qquad \iota:\mathfrak{P}(g)\to\mathfrak{P}(g+1),\quad \iota(\mathfrak{L}) = \mathrm{Comp}(\mathfrak{L},\mathbb{I}_1)\,.$$

To construct the Dyer-Lashof-operation

$$\text{(3.12)}\qquad \tilde{\theta}:S^1\times\mathfrak{P}(g)\times\mathfrak{P}(g')\to\mathfrak{P}(g+g')$$

we can use the composition product. Let $\mathfrak{L}$ have a support of length $b\geq 0$. Then, similiarly to section 2, given u and $v^-<v^+$, let $M(\mathfrak{L},u,v^-,v^+)$ denote that translation and dilation of $\mathfrak{L}$ which has support $[u,u+b]\times[v^-,v^+]$. We need paths $\mathfrak{L}(\alpha)$ and $\mathfrak{L}'(\alpha)$ in $PS\mathbb{C}(g)$ resp. $PS\mathbb{C}(g')$, where $0\leq\alpha\leq 8$, which we define as follows.

$$\text{(3.13)}\qquad \begin{aligned}\mathfrak{L}(\alpha) &= M(\mathfrak{L},\alpha b,0,\tfrac{1}{2})\\ \mathfrak{L}'(\alpha) &= M(\mathfrak{L}',0,\tfrac{1}{2},1) \qquad\qquad 0\leq\alpha\leq 1\,;\end{aligned}$$

$$\begin{aligned} \mathfrak{L}(\alpha) &= M(\mathfrak{L}, b, 0, \tfrac{\alpha}{2}) \\ \mathfrak{L}'(\alpha) &= M(\mathfrak{L}', 0, \tfrac{2-\alpha}{2}, 1) \qquad 1 \le \alpha \le 2\,; \end{aligned} \tag{3.14}$$

$$\begin{aligned} \mathfrak{L}(\alpha) &= M(\mathfrak{L}, b, \tfrac{\alpha-2}{2}, 1) \\ \mathfrak{L}'(\alpha) &= M(\mathfrak{L}', 0, 0, \tfrac{4-\alpha}{2}) \qquad 2 \le \alpha \le 3\,; \end{aligned} \tag{3.15}$$

$$\begin{aligned} \mathfrak{L}(\alpha) &= M(\mathfrak{L}, (\alpha-3)b, \tfrac{1}{2}, 1) \\ \mathfrak{L}'(\alpha) &= M(\mathfrak{L}', 0, 0, \tfrac{1}{2}) \qquad 3 \le \alpha \le 4\,; \end{aligned} \tag{3.16}$$

For $4 \le \alpha \le 8$, the paths go through the same four phases, but the rôles of $\mathfrak{L}$ and $\mathfrak{L}'$ are interchanged. The definition of $\tilde{\theta}$ is then

$$\begin{aligned} \tilde{\theta}(\alpha, \mathfrak{L}, \mathfrak{L}') &= \mathrm{Comp}(\mathfrak{L}'(8\alpha), \mathfrak{L}(8\alpha)) \qquad \text{for} \quad 0 \le \alpha \le \tfrac{1}{2}\,, \\ &= \mathrm{Comp}(\mathfrak{L}(8\alpha), \mathfrak{L}'(8\alpha)) \qquad \text{for} \quad \tfrac{1}{2} \le \alpha \le 1\,. \end{aligned} \tag{3.17}$$

In the case $g = g'$, $\tilde{\theta}$ is equivariant with respect to the antipodal action on an S^1 and the switching on $\mathfrak{P}(g)^2$; this defines the map

$$\theta : S^1 \times_{\mathbb{Z}_2} \mathfrak{P}(g)^2 \to \mathfrak{P}(2g)\,. \tag{3.18}$$

As was developed in [Bödigheimer 1990 II], the map $\tilde{\theta}$ is just one of a family $\tilde{C}^r \times \mathfrak{P}(g_1) \times \cdots \times \mathfrak{P}(g_r) \to \mathfrak{P}(g_1 + \cdots + g_r)$ with parameter spaces $\tilde{C}^r$ the ordered configuration spaces of the plane. This family defines what is called a C_2-operad structure on $\mathfrak{P} = \coprod_{g \ge 0} \mathfrak{P}(g)$.

Putting $\alpha = 0, \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$ in $\tilde{\theta}(\alpha, \mathfrak{L}, \mathfrak{L}')$ we obtain $\mathrm{Wh}(\mathfrak{L}, \mathfrak{L}')$, $\mathrm{Comp}(\mathfrak{L}', \mathfrak{L})$, $\mathrm{Wh}(\mathfrak{L}', \mathfrak{L})$ and $\mathrm{Comp}(\mathfrak{L}, \mathfrak{L}')$; thus Wh and Comp are homotopic, and both homotopy-commutative.

The two involutions Rev, Conj: $\mathfrak{P}(g) \to \mathfrak{P}(g)$ have their names from reversion of the flow, resp. conjugating the complex structure. On the actual moduli space $\vec{\mathfrak{M}}(g)$ this would be $\mathrm{Rev}[F, p, \mathfrak{x}] = [F, p, -\mathfrak{x}]$, and $\mathrm{Conj}[F, p, \mathfrak{x}] = [\bar{F}, p, \mathfrak{x}]$, where $\bar{F}$ denotes the conjugate complex structure on the surface F. The first reverses the flow and it is somewhat complicated to write down the new slits. This involution is actually part of an S^1-action on $\vec{\mathfrak{M}}(g)$ — and thus on $\mathfrak{P}(g)$ — rotating the tangent direction $\mathfrak{x}$. The conjugation corresponds to taking the complex-conjugate of $\mathfrak{L}$; in normalized coordinates this is

$$\begin{aligned} &\mathrm{Conj}[(x_0, y_0), \ldots, (x_{4g-1}, y_{4g-1}) | \lambda] = \\ &\qquad [(x_{4g-1}, 1 - y_{4g-1}), \ldots, (x_0, 1 - y_0) | \omega \circ \lambda \circ \omega^{-1}]\,, \end{aligned} \tag{3.19}$$

with $\omega \in \mathfrak{S}_{4g-1}$ as in (2.20).

4. First-return maps.

It is now almost evident how to define the map

$$\Phi : \vec{\mathfrak{M}}(g) \cong \mathfrak{P}(g) \to \mathfrak{Ex}(4g) \tag{4.1}$$

by using the parametrization of the moduli space as the space of normalized parallel slit domains. We set

$$\Phi[L_0, \dots, L_{4g-1}|\lambda] = [y_0, \dots, y_{4g-1}|\pi] \tag{4.2}$$

when (x_k, y_k) is the endpoint of the slit L_k, and $\pi \in \mathfrak{S}_{4g-1}$ is defined by $\pi(0) := \lambda(0)$, and $\pi(k) := \lambda(\pi(k-1)+1)$ for $k = 1, \dots 4g-2$. Condition (3.5) implies that π is bijective on the indices $0, 1, \dots 4g-2$, since $\pi(k) = \sigma(\pi(k-1)) = \sigma^{k+1}(-1)$. (One can extend the definition of π by $\pi(4g-1) = \pi(-1) = -1$, to be understood mod $4g$.)

Φ is well-defined, since a jump as in (3.7) will only move a trivial interval around as in (1.2) and (1.3). Thus Φ is continuous. The normalizations correspond, so the Φ takes $\mathfrak{P}'(g)$ to $\mathfrak{Ex}'(4g)$, and $\mathfrak{P}''(g) = PS\mathbb{C}(g)$ to $\mathfrak{Ex}''(g)$.

Next we want to give a geometric description of $\Phi[L]$, namely as a first-return-map of some flow on $F(L)$. The additional feature of a directed Riemann surface $[F, p, \mathfrak{x}]$ is the existence of a dipole flow on F. This is the gradient flow of a harmonic function $U : F \to \mathbb{R} \cup \infty$ determined by the conformal structure of F and the properties:

(4.3) (1) p is the only singularity of U,
(2) for a local parameter z around p such that $z(p) = 0$ and $dz(\mathfrak{x}) = (dx)$ the local expression of U is $U(z) = Re(\frac{1}{z})+$ regular terms,

Up to an additive real integration constant b_1 and a multiplicative real constant $a > 0$ such a function U is unique. One can find such a U by minimizing the Dirichlet integral of $U(z) - Re(\frac{1}{z})$. Another way is to take the unique abelian differential with a double pole at p in the direction $\mathfrak{x}$ which has pure-imaginary periods; its real part can be integrated to obtain a harmonic function.

The unstable submanifolds of the gradient flow $\varphi = \text{grad}(U)$ form the critical graph in F, whose complement F_0 is simply-connected; therefore U is the real part of some injective holomorphic function $W = U + iV : F_0 \to \mathbb{C}$; W is unique up to another additive integration constant b_2 for the harmonic conjugate V. The complement of $W(F_0)$ in $\mathbb{C}$ are the slits $L_0 \cup L_1 \cup \cdots \cup L_{4g-1}$ of the parallel slit domain $\mathfrak{L}$ representing $[F, p, \mathfrak{x}]$. On $W(F_0)$ the flow φ becomes the horizontal flow $\varphi = -dx$, since U becomes the function x.

Apart from the dipole p there are no other sinks or sources for this flow, and (counted with multiplicities) there are $2g$ zeroes. Let Q be a quadrilateral around p such that one pair of sides are integral curves, the other pair are equipotential curves, and no zero is in its interior. The W-image of its complement is a rectangle in $\mathbb{C}$ with sides parallel to x, y-axes that contains all slit endpoints; the interior of the largest such Q is the complement of the support of $\mathfrak{L}$. Assuming everything to be normalized $W(Q)$ is the complement of the (open) unit square.

The flow φ induces a function f from the right vertical $x = 1$ to the left vertical $x = 0$, defined for all $0 \leq y \leq 1$, which are not the levels $y_0, \dots, y_{4g-1}$ of zeroes of φ. For example, the interval $Y_0 = [y_0, y_1]$ between the first two levels y_0 and y_1, is moved by the flow φ horizontally across the support to the left vertical $x = 0$. But here—after regluing—the number of this interval becomes $\pi^{-1}(0)$. Thus we see that the induced function f is the inverse of the exchange map $f_{\mathfrak{y}}$ for $\mathfrak{y} = \Phi(\mathfrak{L})$.

The map Φ is not surjective, because the conditions (3.1, 2, 5) on λ impose conditions on π. From the definition of π we have $\pi = \lambda \circ \rho \circ \pi \circ \rho^{-1}$, if $\rho \in \mathfrak{S}_{4g}$ is the cyclic permutation. In other words, $\lambda = \pi \circ \rho \circ \pi^{-1} \circ \rho^{-1}$ is the commutator of π with this particular element ρ, and has to be of order 2 according to (3.2). For example, for $g = 1$ there are 6 2-cells, 2 1-cells and 1 0-cell in $\mathfrak{Ex}(4)$; but only one of the 2-cells and one of the 1-cells make up the image of Φ.

The various structures on $\vec{\mathfrak{M}}(g)$ and $\mathfrak{Ex}(4g)$ are preserved by Φ, what we summarize in

(4.4) **Proposition.** *The map* $\Phi : \vec{\mathfrak{M}} \to \mathfrak{Ex}(4g)$ *has the following properties:*

(i) $\Phi \circ \mathit{Stab} = \iota^3 \circ \mathit{Stab} \circ \Phi$,
(ii) $\Phi \circ \mathit{Wh} = \mathit{Wh} \circ (\Phi \times \Phi)$,
(iii) $\Phi \circ \mathit{Comp} = \mathit{Comp} \circ (\Phi \times \Phi)$,
(iv) $\Phi \circ \tilde{\theta} = \tilde{\theta} \circ (\mathit{id} \times \Phi \times \Phi)$, *and* $\Phi \circ \theta = \theta \circ (\mathit{id} \times \Phi)$,
(v) $\Phi \circ \mathit{Rev} = \mathit{Rev} \circ \Phi$,
(vi) $\Phi \circ \mathit{Conj} = \mathit{Conj} \circ \Phi$.

The notation is set up so that the proofs are straight-forward.

Although Φ is not surjective, it is "stably surjective" in the following sense: for any $\mathfrak{y} \in \mathfrak{Ex}(n)$ there is an $\mathfrak{L} \in \mathfrak{P}(g)$ for some $g \leq n$, such that $\Phi(\mathfrak{L}) = \mathfrak{y}$ in $\mathfrak{Ex}(n')$, $n' \geq n$, $4g$, where we suppressed the inclusions in the notation.

To construct such an $\mathfrak{L}$ let $m-1 \leq n-1$ be the number of non-trivial intervals of $\mathfrak{y}$, i.e. we can assume that $\mathfrak{y} \in \mathfrak{Ex}(m)$ and all $t_i > 0$, $i = 0, 1, \dots, m-2$. First, put m slits at the origin,

$$L_i \text{ at } (0,0) \quad \text{for} \quad i = 0, 1, \dots, m-2\,.$$

Next, for each interval $Y_0, Y_1, \dots, Y_{m-2}$ put a slit at $(1, y_k)$, at $(0, y_k)$ and at $(1, y_{k+1})$, i.e.

$$L_1 \text{ at } (1, y_k) \qquad \text{for} \quad i = m - 1 + 3k + 1\,,$$

$$L_1 \text{ at } (0, y_k) \qquad \text{for} \quad i = m - 1 + 3k + 2\,,$$

$$L_1 \text{ at } (1, y_{k+1}) \quad \text{for} \quad i = m - 1 + 3k + 3\,,$$

when $k = 0, 1, \dots, m-2$. The pairing is set as

$$\lambda(k) = m - 1 + 3\pi(k) + 2, \quad \text{for } k = 0, 1, \dots, m-2,$$

$$\lambda(m - 1 + 3k + 1) = \lambda(m - 1 + 3k + 3), \quad \text{for } k = 0, 1, \dots, m-2\,.$$

This $\mathfrak{L}$ will lie in $\mathfrak{P}(g)$ for $g = m - 1$, since all $t_i > 0$.

5. Homotopy type of $\mathfrak{Cr}(n)$.

As a result of the rather crude topology we have chosen for $\mathfrak{Cr}(n)$ it is now easy to determine its homotopy-type.

(5.1) **Proposition.** *$\mathfrak{Cr}(n)$ is homotopy-equivalent to a bouquet of e_n spheres of dimension $n-2$, where* $e_n = \sum_{k=1}^{n-2}(-1)^{n+1}(k+1)!$.

Here we set $e_2 = 1$. Recursively we have $e_{n+1} = n! - e_n$ for $n \geq 3$. Thus, for example, $e_3 = 2$, $e_4 = 4$, $e_5 = 20$, $e_6 = 100$, $e_7 = 620$ and $e_8 = 4420$.

To prove the above we use the skeletal filtration of the space $\mathfrak{Cr}(n+1)$ and the inclusion $\iota : \mathfrak{Cr}(n) \to \mathfrak{Cr}(n+1)$. Using (1.2) and (1.3) one can move all intervals Y_i with $t_i = 0$ to the right, and also move all intervals X_j with $t_{\pi(j)} = 0$ to the right, thus bringing $\mathfrak{y}$ into a normal form. This shows that the image of ι is precisely the $(n-2)$-skeleton of $\mathfrak{Cr}(n+1)$. Furthermore, ι is homotopic to the constant map by the homotopy

(5.2) $\iota_s[y_0, y_1, \dots, y_{n-1}|\pi] = [(1-s)y_0, (1-s)y_1, \dots, (1-s)y_{n-1}, 1|\pi \oplus id]$

with $0 \leq s \leq 1$. $\iota = \iota_0$ adds a trivial interval at the right end, which is increased throughout the homotopy at the expense of all others; under ι_1 those end up as trivial intervals at the left end; but $\iota_1[y]$ is the basepoint in $\mathfrak{Cr}(n+1)$ because of (1.2) and (1.3).

Since $\mathfrak{Cr}(2)$ is just a point, it follows by induction that $\mathfrak{Cr}(n)$ is a bouquet of spheres, all of the same dimension $n-2$. To determine their number e_n we only need to compare the Euler characteristic of $\mathfrak{Cr}(n)$ and of $\bigvee^{e_n} S^{n-2}$.

Counting the normal forms of k-cells in $\mathfrak{Cr}(n)$ we have

$$\chi(\mathfrak{Cr}(n)) = \sum_{k=0}^{n-2}(-1)^k(k+1)! \,. \tag{5.3}$$

On the other hand we obtain

$$\chi\Big(\bigvee^{e_n} S^{n-2}\Big) = 1 + (-1)^n e_n \,. \tag{5.4}$$

The formula for e_n follows. ∎

6. Groups of exchanges.

The inclusions $\iota : \mathfrak{Cr}(n) \to \mathfrak{Cr}(n+1)$ lead to an infinite-dimensional complex

$$\mathfrak{Cr}(\infty) = \varinjlim_{\iota} \mathfrak{Cr}(n) = \bigcup_{n \geq 2} \mathfrak{Cr}(n) \,. \tag{6.1}$$

A "stable" exchange $\mathfrak{y} \in \mathfrak{Ex}(\infty)$ disregards all non-trivial intervals. The space $\mathfrak{Ex}(\infty)$ is contractible by (5.1), and also inherits the strictly-associative multiplication Comp (apart from the H-space structure via Wh), with unit the stable class of $\mathbb{I}_1$. But it is not a group, because we still keep track of superfluous cutpoints. To remedy this we introduce a further equivalence relation on unnormalized exchanges.

$$[y_0, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_{n-1}|\pi] \approx [y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_{n-1}|\hat{\pi}] \tag{6.2}$$

if $i = \pi(j)$ and $i+1 = \pi(j+1)$, and where $\hat{\pi} \in \mathfrak{S}_{n-2}$ is the permutation $\hat{\pi}(k) = \pi(k)$ for $k = 0, 1, \dots j$, and $\hat{\pi}(k) = \pi(k+1) - 1$ for $k = j+2, \dots, n-3$. Denote equivalence classes by $[[y]]$.

The points of the quotient space $\mathcal{E}(n) = \mathfrak{Ex}''(n)/\approx$ now correspond to the L^2-classes of functions $f : \mathbb{R} \to \mathbb{R}$, which have at most n points of discontinuity, and are isometric and orientation-preserving on all intervals between those points. Note that on $\mathcal{E}(n)$ the support (i.e. left and right endpoint) is no longer a continuous function of the exchange. The inclusions ι induce inclusions $\iota : \mathcal{E}(n) \to \mathcal{E}(n+1)$, and thus we have a "stable" complex

$$\mathcal{E}(\infty) = \varinjlim_{\iota} \mathcal{E}(n) = \bigcup_{n \geq 2} \mathcal{E}(n)\,. \tag{6.3}$$

The composition product makes $\mathcal{E}(\infty)$ into a group. We claim that $\mathcal{E}(\infty)$ is contractible. For this we use a section of the projection $\mathfrak{Ex}(n) \to \mathcal{E}(n)$, which sends $[f] \in \mathcal{E}(n)$ to the following exchange $\mathfrak{y}$. If $\tilde{y}_0 < \tilde{y}_1 < \dots < \tilde{y}_{m-1}$ are the $m \leq n$ points of discontinuity, then the intervals $]\tilde{y}_j, \tilde{y}_{j+1}[$ are mapped to some open intervals with intermediate points $y_0 < y_1 < \dots < y_{m-1}$. If $f(]\tilde{y}_j, \tilde{y}_{j+1}[) =]y_i, y_{i+1}[$, then set $\pi(j) = i$, and $\mathfrak{y} = [y_0, \dots, y_{m-1}|\pi]$. This is a well-defined (and thus continuous) section; it induces a section to $\mathfrak{Ex}''(\infty) = \bigcup_{n \geq 2} \mathfrak{Ex}''(n) \to \mathcal{E}(\infty)$. The contractible space $\mathfrak{Ex}''(\infty) \cong \mathbb{R} \times \mathbb{R}_+ \times \mathfrak{Ex}(\infty)$ contains $\mathcal{E}(\infty)$ as a retract, thus it is itself contractible.

The group $\mathcal{E}(\infty)$ contains the infinite symmetric group $\mathfrak{S}_\infty = \varinjlim \mathfrak{S}_n$ as a discrete subgroup, by sending $\pi \in \mathfrak{S}_{n-1}$ to $[[0, 1, 2, \dots, n-1|\pi]]$ in $\mathcal{E}(n)$. The homogeneous space $\mathcal{E}(\infty)/\mathfrak{S}_\infty$ is then a curious model for the classifying space $B\mathfrak{S}_\infty$.

An action of the finite symmetric group $\mathfrak{S}_{n-1}$ on $\mathfrak{Ex}''(n)$, which can be defined by $[y_0, \dots |\pi].\alpha = [y_0, \dots |\pi \circ \alpha]$, is not free. Its isotropy subgroup on a cell of codimension k consists of all elements conjugate to some element in $\mathfrak{S}_k \leq \mathfrak{S}_{n-1}$.

It seems obvious that there is a braid version of the spaces $\mathfrak{Ex}(n)$, in which the braid group $\mathfrak{B}_{n-1}$ replaces the symmetric group $\mathfrak{S}_{n-1}$. Some braids are a 3-dimensional phenomenon, those spaces should be connected to directed Riemann surfaces embedded in $\mathbb{R}^3$. One could then speculate whether there is some connection to the work of [Greenberg-Sergiescu 1991] which relates groups of piece-wise linear homeomorphisms of $[0, 1]$ to the classifying space of the infinite braid group.

There is an interesting function $D : \mathfrak{Ex}''(\infty) \to \mathbb{R}$, which descends to the group $\mathcal{E}(\infty)$. For an exchange $\mathfrak{y} = [y_0, \dots, y_{n-1}|\pi]$ we set $w_i = \sum_{k=0}^{\pi(i)-1} t_k - \sum_{k=0}^{i-1} t_{\pi(k)}$, and then define

$$D(\mathfrak{y}) = \sum_{i=0}^{n-2} t_{\pi(i)} \cdot |w_i| \,. \tag{6.4}$$

w_i is the displacement of the interval $\tilde{Y}_i$ under $f_{\mathfrak{y}}$; so D measures the work involved in moving the intervals $\tilde{Y}_i$ around. Since w_i is linear in the barycentric coordinates, D is a quadratic form, just depending on π. (Note: $\sum t_{\pi(i)} w_i = 0$, a kind of energy preservation.) D is well-defined, since trivial intervals do not contribute to the sum; and superfluous cut points only decompose one summand into two, without changing the sum. D is related to the Sah-Arnoux-Fahti invariant, which takes values in the second exterior power $\bigwedge^2_{\mathbb{Q}}(\mathbb{R})$ of $\mathbb{R}$ over the rationals; see [Arnoux 1981], [Sah 1979], [Veech 1984].

D has the following properties, where we used $\circ$ to denote the composition product Comp :

$$\begin{aligned} &(1) \quad D(\mathfrak{y} + b) = D(\mathfrak{y}), \\ &(2) \quad D(\mathfrak{y}.a) = a^2 D(\mathfrak{y}), \\ &(3) \quad D(\mathfrak{y}) \geq 0, \text{ and } D(\mathfrak{y}) = 0 \text{ if and only if } f_{\mathfrak{y}} = \text{ id}, \\ &(4) \quad D(\mathfrak{y} \circ \mathfrak{y}') \leq D(\mathfrak{y}) + D(\mathfrak{y}') \,. \end{aligned} \tag{6.5}$$

If $\mathfrak{Ex}(n)$ were a manifold, D might be a candidate for an interesting Morse function.

7. Remarks.

We have seen that the spaces $\mathfrak{Ex}(n)$ of exchanges are rather simple from the homological point view, since all their homology is concentrated in one dimension. This implies, of course, that the structural maps described in section 4 are all null-homotopic.

Responsible for this unfortunate situation are two flaws: (1) we included far too many cells; (2) the topology we defined on $\mathfrak{Ex}(n)$ is far too coarse.

To address (1), recall that Φ is not surjective; in fact, its image intersects a rather small number of top dimensional cells, because very few permutations π arise from some λ in the way of (4.2). For example, if $g = 1$ resp. 2 resp. 3, then 1 out of 6, resp. 21 out of $7! = 5040$, resp. 1485 out of $11! = 39916800$ open cells of maximal dimension $4g-2$ lie in $im(\Phi)$. Moreover, $\mathfrak{Ex}(n)$ is a compact complex including all lower dimensional faces of cells, whereas $\mathfrak{P}(g)$ is an open manifold excluding many faces of cells which would contain only degenerate surfaces. We have no good description of the image of Φ.

For (2), the topology we chose seems to be the natural one, unless one restricts the type of permutations π — which brings us back to (1).

It seems that the right thing to do is to restrict attention to only those π arising from pairings $\lambda \in \mathfrak{S}_{4g}$ which satisfy (3.1, 2, 5). We keep λ in the notation and call $y = (y_0, \dots, y_{4g-1}|\lambda)$ an interval recombination; it cuts the real line $\mathbb{R}$ at pairs of points y_i and $y_{\lambda(i)}$, gluing the left resp. right side of y_i to the right resp. left side of $y_{\lambda(i)}$, just as we did with slits when creating the surface $F(L)$ in section 3. The equivalence relation is now suggested by the jumps of slits in (3.7), disregarding the condition $x_{k-1} \leq x_k$.

This space, denoted by $\mathfrak{Rec}(g)$, is a finite complex with a forgetful map to $\mathfrak{Ex}(4g)$. It has all the structure of $\mathfrak{P}(g)$, and comes with a structure preserving map $\Phi : \mathfrak{P}(g) \to \mathfrak{Rec}(g)$, defined as $\Phi[(x_0, y_0), \dots |\lambda] = [y_0, \dots |\lambda]$.

We will return to this space, since it turns out to be homology-equivalent to an interesting compactification of the moduli space $\vec{\mathfrak{M}}(g)$.

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