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Relative Trace Formulae in Analytic Number Theory

Set 1: A relative trace formula for Jacobi forms

Exercise: Derive a (spectral) summation formula for the Fourier coefficients of Jacobi Forms, which is analogous to the classical Petersson formula. This can be done using Poinncaré series or with the relative trace formula approach.

Challenge: Does a formula similar to the Kuznetsov formula exist for the Jacobi group? It should be possible to set this up using the relative trace formula approach. All the necessary preliminaries are for example in the book *Elements of the Representation Theory of the Jacobi Group* by R. Berndt and R. Schmidt. To the best of my knowledge such a formula is not yet in the literature.

Some notation: We write $\mathcal{J}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \cdot S^1$ for the Jacobi Group.[1] For $A \in \mathrm{SL}_2(\mathbb{R})$, $\mathbf{x} \in \mathbb{R}^2$ and $\zeta \in S^1$ we write $\xi = (A, \mathbf{x}, \zeta) \in \mathcal{J}(\mathbb{R})$. Multiplication is given by

$$(A_1, \mathbf{x}_1, \zeta_1) \cdot (A_2, \mathbf{x}_2, \zeta_2) = (A_1A_2, \mathbf{x}_1A_2 + \mathbf{x}_2, \zeta_1\zeta_2\varepsilon) \text{ for } \varepsilon = e\left(\det\begin{pmatrix}\mathbf{x}_1A_2\\ \mathbf{x}_2\end{pmatrix}\right).$$

Every element $\xi \in \mathcal{J}(\mathbb{R})$ can be written as

$$\xi = (A, 0, 1) \cdot (1, \mathbf{x}, 1) \cdot (1, 0, \zeta) =: A[\mathbf{x}]\zeta.$$

We put $\Gamma^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \cdot \{1\} \subseteq \mathcal{J}(\mathbb{R})$. An important subgroup is

$$\Gamma^J_\infty = \{(A, \mathbf{x}, 1) \in \Gamma^J : \mathbf{x} = (0, x_2) \text{ and } A = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\}.$$

The Jacobi group acts on $\mathbb{H} \times \mathbb{C}$ by

$$A[\mathbf{x}]\zeta.(\tau, z) = (\frac{a\tau + b}{c\tau + d}, \frac{z + x_1\tau + x_2}{c\tau + d}) \text{ where } A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \text{ and } \mathbf{x} = (x_1, x_2).$$

We will also write $A\tau$ for the action in the first coordinate. This is nothing but the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ via Möbius transformations. For functions $\phi\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ and integers m, k we define the slash operator

$$[\phi|_{k,m}A[\mathbf{x}]\zeta](\tau, z) = (c\tau + d)^{-k} e\left(-\frac{c(z + x_1\tau + x_2)^2}{c\tau + d} + x_1^2\tau + 2x_1z + x_1x_2\right)^m \zeta^m \phi(A\tau, \frac{z + x_1\tau + x_2}{c\tau + d}).$$

A Jacobi form of weight k and index m is a complex valued function $f\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying

- $(f|_{k,m}\gamma)(\tau, z)\colon = f(\tau, z)$ for all $\gamma \in \Gamma^J$;

[1] This group appears in many disguises. We chose this for simplicity.

- f is holomorphic;
- f has a Fourier series of the form

$$f(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4mn-r^2\geq 0}} c_f(n,r)e(n\tau + rz).$$

Further, we call f a cusp form if $c_f(n,r) = 0$ unless $4mn - r^2 > 0$. The space of all Jacobi forms is denoted by $J_{k,m}$. It features the subspace of cuspidal Jacobi forms $J_{k,m}^{\text{cusp}}$. Finally we introduce the appropriate inner product on $J_{k,m}^{\text{cusp}}$. We usually write $\tau = u + iv \in \mathbb{H}$ and $z = x + iy \in \mathbb{C}$. Then we define

$$\mu_{k,m}(\tau, z) = v^{\frac{k}{2}} e^{-2\pi \frac{y^2}{v}}.$$

The Petersson inner product is given by

$$\langle f, g\rangle = \int_{\Gamma^J\backslash(\mathbb{H}\times\mathbb{C})} f\overline{g} \cdot \mu_{k,m}^2 dV \text{ for } dV = v^{-3}dudvdxdy.$$

It is a nice exercise to check that this is well defined.

Possibly relevant literature:

- **For Poincaré series:** *Heegner Points and Derivatives of L-Series. II* by B. Gross, W. Kohnen and D. Zagier (1987).
- **For the Bergman kernel:** *A Trace Formula for Jacobi Forms* by D. Zagier (1989).
- **For the adelic picture:** *Elements of the Representation Theory of the Jacobi Group* by R. Berndt and R. Schmidt (1998).