Kakeya sets over finite fields

Basic Notions Seminar

Joris Roos

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1 Kakeya sets in $\mathbf{R}^n$

Definition 1. A *Kakeya set* $E \subset \mathbf{R}^n$ is a compact set containing a unit line segment in every direction.

The existence of Kakeya sets of arbitrarily small Lebesgue measure and even of measure 0 is a well-known, but counter-intuitive fact, which is often discussed in beginners courses on measure theory[1].
It is therefore a natural question to ask for the (fractional) dimension of this set. For sake of simplicity, let us consider the Minkowski dimension.

Definition 2. Let $E \subset \mathbf{R}^n$ be measurable and $N_\delta(E) = \{x \in \mathbf{R}^n : \mathrm{dist}(x, E) \le \delta\}$ for $\delta > 0$. If the limit

$$\lim_{\delta \to 0} \left(n - \frac{\log |N_\delta(E)|}{\log \delta} \right)$$

exists, it is called the *Minkowski dimension* of E.

Then the simplest form of the Kakeya conjecture is as follows.

Conjecture 1 (Kakeya)**.** *A Kakeya set in* $\mathbf{R}^n$ *has Minkowski dimension* n.

This seemingly simple proposition has turned out to be a notoriously difficult problem over the course of 20th century mathematics and it still remains an enigma for $n \ge 3$. It has generated vast amounts of mathematics and is deeply intertwined with numerous other open questions, most prominently the restriction and Bochner-Riesz conjectures in harmonic analysis.
Despite decades of effort, volume-filling bulks of research articles written on the topic and dozens of approaches taken, the best we can say today for large n is that the Minkowski

[1] Often also called Besicovitch sets.

dimension is at least $\frac{n}{\alpha} + c$ where α is a known, fixed number between one and two[2](Katz, Tao [3], [4]) and c a known positive constant smaller than one.

2 Finite fields

Let us fix $\mathbf{F}$ to be a finite field with q elements. Changing the ground field in the above to be $\mathbf{F}$ instead of $\mathbf{R}$ gives a new perspective on Kakeya sets.

Definition 3. We call $K \subset \mathbf{F}^n$ a *Kakeya set*, if for all $y \in \mathbf{F}^n$ there exists $x \in \mathbf{F}^n$ such that the line $\{x + y \cdot t \,:\, t \in \mathbf{F}\}$ is contained in K.

The appropriate analogue of the Kakeya conjecture, first suggested by Wolff [5], is as follows.

Conjecture 2 (Finite field Kakeya)**.** *For $n \in \mathbf{N}$ there exists a constant $c = c_n > 0$ such that every Kakeya set in $\mathbf{F}^n$ has at least cardinality $c_n q^n$.*

This conjecture has been turned into a theorem by Z. Dvir [2] in 2008. The proof is astonishingly elementary and short (however the margin here is, of course, too small to contain it, so I postpone its presentation until the seminar as a motivation for attendance). It proceeds roughly by counting zeros of multivariate polynomials over $\mathbf{F}$. Methods motivated by Dvir's proof are often subsumed under the catchy term *the polynomial method.* Various attempts have been made to transfer these results to $\mathbf{R}^n$. For instance, Guth and Carbery, Valdimarsson [1] have recently applied the polynomial method to certain multilinear analogues of the Kakeya conjecture posed by Bennett, Carbery and Tao.

References

[1] Anthony Carbery and Stefán Ingi Valdimarrson. The endpoint multilinear Kakeya theorem via the Borsuk-Ulam theorem. *J. Funct. Anal.*, 264(7):1643–1663, 2013.

[2] Zeev Dvir. On the size of Kakeya sets in finite fields. *J. Amer. Math. Soc.*, 22(4):1093–1097, 2009.

[3] Nets Hawk Katz and Terence Tao. Bounds on arithmetic projections, and applications to the Kakeya conjecture. *Math. Res. Lett.*, 6(5-6):625–630, 1999.

[4] Nets Hawk Katz and Terence Tao. New bounds for Kakeya problems. *J. Anal. Math.*, 87:231–263, 2002.

[5] Thomas Wolff. Recent work connected with the Kakeya problem. *Prospects In Mathematics*, 1999.

[2] $\alpha = 1.67513..$